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NEW OSCILLATION CRITERIA FOR SECOND ORDER HALF-LINEAR NEUTRAL TYPE DIFFERENCE EQUATION WITH DISTRIBUTED DEVIATING ARGUMENTS

ABSTRACT In this paper, we will study the oscillatory properties of the second order half-linear difference equationwith distributed deviating arguments. We obtain several new sufficient conditions for the oscillation of all solutions of this equation. Our results improve and extend some known results in the literature. Examples which dwell upon the importance of our results are also included.

Keywords:-Difference equation, Oscillatory solutions, neutral, deviating arguments.

2000 AMS Mathematics Subject Classification: **39A10, 39A12**.

INTRDUCTION

In recent years, there has been an increasing interest in the study of the oscillatory behavior of solutions of difference equations (see, e.g., [l-19] and the references cited therein). In this paper, we are concerned with the oscillatory behavior of solutions of second -order half-linear neutral typedifference equationwith distributed deviating arguments of the form

$$\Delta\left(a_n\psi(x_n)|\Delta(x_n+p_nx_{n-\tau})|^{\alpha-1}\Delta(x_n+p_nx_{n-\tau})\right)+\sum_{\xi=a}^{b}q(n,\xi)f\left(x\left(g(n,\xi)\right)\right)=0. \qquad (1.1)$$

Where $\tau$is nonnegative integers, N = {1,2,...},Δdenotes the forward difference operator defined by $\Delta x_n = x_{n+1}$ - x_n.By a solution of equation (1.1) we mean a nontrivial sequence $\{x_n\}$defined on $N(n_0)$, which satisfying equation (1.1)for all $n \geq n_0$. A solution $\{x_n\}$ of equation (1.1)is said to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise. Equation(1.1)is called oscillatory if all its solutions are oscillatory. Throughout this paper, we will assume the following hypotheses:

(A_1)$p_n$ispositive, $0 \leq p_n \leq p \leq +\infty for\ n = 0,1,2,...$,where$p$is a constant.

$$(A_2)\ a_n > 0\,, \psi: R \to (0,\infty), n = 0,1,2,\dots \text{ such that } \sum_{n=n_0}^{\infty}\frac{1}{\left(a_n\psi(x_n)\right)^{\frac{1}{\alpha}}} = \infty.$$

(A_3) $q(n,\xi) > 0$ on $N(n_0)\times N(a,b)$ and $g: N(n_0)\times N(a,b) \to N$satisfies $n \geq g(n,\xi) for\ \xi \in N(a,b)$ and $\lim_{n\to\infty} \min g(n,\xi) = \infty$.

(A_4) $f \in C(R,R)$such that$xf(x) > 0\ for x \neq 0$and$f(u)/u^{\alpha} \geq K > 0$.

The paper is organized as follows. In Section 2, we will state and prove the main oscillation theorems and in Section 3, we provide some examples to illustrate the main results.

MAIN RESULTS

In this section, we establish some new oscillation criteria for the equation (1.1). We beginwith some useful lemmas, which will be used later.

Lemma 2.1. Let $\{x_n\}$be a nonoscillatory solution of equation (1.1).Then there exists a $n \geq n_0$ such that

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$z_n \geq 0$, $\Delta z_n \geq 0$ and $\Delta(a_n\psi(x_n)|\Delta z_n|^{\alpha-1}\Delta z_n) \leq 0$ for $n \geq n_0$. (2.1)

proof. Let $\{x_n\}$ is eventually positive solution of equation(1.1), we may assume that $x_n > 0$, $x_{n-\tau} > 0$, and $x(g(n,\xi)) > 0$ for $n \geq n_0$, $\xi \in (a,b)$. Set $z_n = x_n + p_n x_{n-\tau}$. By, assumption($A_1$), we have $z_n > 0$, and from equation(1.1), we get

$$\Delta(a_n\psi(x_n)|\Delta z_n|^{\alpha-1}\Delta z_n) = -\sum_{\xi=a}^{b} q(n,\xi) f\left(x\big(g(n,\xi)\big)\right) \leq 0. \quad (2.2)$$

Therefore, $\{a_n\psi(x_n)|\Delta z_n|^{\alpha-1}\Delta z_n\}$ is non-increasing sequence. Now we have two possible cases for Δz_n either $\Delta z_n < 0$ eventually or $\Delta z_n > 0$ eventually. Suppose that $\Delta z_n < 0$ for $n \geq n_0$. Then from (2.2), there is an integer $m \geq n$ such that $\Delta z_m < 0$ and

$$a_n\psi(x_n)(\Delta z_n)^\alpha \leq a_m\psi(x_m)(\Delta z_m)^\alpha, \qquad for\, n \geq m. \quad (2.3)$$

Dividingby $a_n\psi(x_n)$ and summing the last inequality from m to $k-1$, we obtain

$$z_k \leq z_m + \big(a_m\psi(x_m)\big)^{\frac{1}{\alpha}}\Delta z_m \sum_{n=m}^{k-1} \frac{1}{\big(a_n\psi(x_n)\big)^{\frac{1}{\alpha}}} \qquad for\ \ k \geq m.$$

This implies that $z_k \to -\infty$ as $k \to \infty$, which is a contradiction the fact that z_n is positive. Then $\Delta z_n > 0$. This completes the proof of Lemma 2.1. ■

Lemma 2.2. Assume that $\alpha \geq 1, x_1, x_2 \in [0,\infty)$. Then

$$x_1^\alpha + x_2^\alpha \geq \frac{1}{2^{\alpha-1}}(x_1 + x_2)^\alpha.$$

proof. The proof can be found in [6, pp. 292] and also in [8, Remark 2.1].

Lemma 2.3. Assume that $0 < \alpha \leq 1, x_1, x_2 \in [0,\infty)$. Then

$$x_1^\alpha + x_2^\alpha \geq (x_1 + x_2)^\alpha. \quad (2.4)$$

proof. Assume that $x_1 = 0$ or $x_2 = 0$. Then we have (2.4). Assume that $x_1 > 0$ and $x_2 > 0$ .Define $f(x_1,x_2) = x_1^\alpha + x_2^\alpha - (x_1+x_2)^\alpha$. Fix x_1. Then

$$\frac{df(x_1,x_2)}{dx_2} = \alpha x_2^{\alpha-1} - \alpha(x_1+x_2)^{\alpha-1}$$

$$= \alpha[x_2^{\alpha-1} - (x_1+x_2)^{\alpha-1}] \geq 0, \text{since } 0 < \alpha \leq 1.$$

Thus, f is nondecreasing with respect to x_2, which yields $f(x_1,x_2) \geq 0$. This completes theproof.

Lemma 2.4. Let $\alpha > 0$ If $f_n > 0$ and $\Delta f_n > 0$ for all $n \geq n_0 \in N$, then

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$$\Delta f_n^{\alpha} \geq \alpha f_n^{\alpha-1} \Delta f_n \;\; if \quad \alpha \geq 1,$$

and

$$\Delta f_n^{\alpha} \geq \alpha f_{n+1}^{\alpha-1} \Delta f_n \;\; if \quad 0 < \alpha \leq 1,$$

for all $n \geq n_0$.

proof. By Mean value theorem, we have for $n \geq n_0$

$$\Delta f_n^{\alpha} = f_{n+1}^{\alpha} - f_n^{\alpha} = \alpha t^{\alpha-1} \Delta f_n$$

where$f_n < t < f_{n+1}$. The result follows by taking $t > f_n$when $\alpha \geq 1$and $t < fn + 1$when$0 < \alpha \leq 1$.

Throughout this subsection we assume that there exists a double sequence $\{H_{m,n} | m \geq n \geq 0\}$ and$h_{m,n}$ such that

$$\text{(i)} H_{m,m} = 0 \; for \; m \geq 0,$$

$(ii) H_{m,n} > 0 \; for \; m > n > 0,$

$(iii) \Delta_2 H_{m,n} = H_{m,n+1} - H_{m,n} \leq 0 \; for \; m > n \geq 0,$

$$\text{(iv)} h_{m,n} = -\frac{\Delta_2 H_{m,n}}{\sqrt{H_{m,n}}}.$$

In the following results, we shall use the following notation

$$R_n := \frac{1}{\left(a(G_n)\psi\left(x(G_n)\right)\right)^{1/\alpha}}, \Theta_n := \rho_n \frac{R_n}{\rho_{n+1}^{1+\frac{1}{\alpha}}}, \varphi_n := 2^{1-\alpha} \frac{\rho_n J_n}{\rho_{n+1}^2}, \vartheta_{m,n} := \left(\frac{\Delta\rho_n}{\rho_{n+1}} - \frac{h_{m,n}}{\sqrt{H_{m,n}}}\right).$$

$$\Psi_n := K\rho_n \delta_{n,\xi} - \rho_n \Delta(a_n \psi(x_n) \beta_n) + \rho_n R_n (a_{n+1} \psi(x_{n+1}) \beta_{n+1})^{1+\frac{1}{\alpha}}.$$

$$\eta_n := \Delta\rho_n + \alpha \rho_n R_n \left(1 + \frac{1}{\alpha}\right) (a_{n+1} \psi(x_{n+1}) \beta_{n+1})^{\frac{1}{\alpha}}.$$

$$\mu_{m,k} := \frac{\rho_{n+1}^{\alpha+1}}{(1+\alpha)^{1+\alpha}} \frac{\left(\frac{\eta_n H_{m,n}}{\rho_{n+1}} - h_{m,n}\sqrt{H_{m,n}}\right)^{1+\alpha}}{\left(\rho_n R_n H_{m,n}\right)^{\alpha}}.$$

Next, we state and prove the main theorems.

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Theorem 2.1.Let $\alpha \geq 1$. Further, assume that there exists a positive non decreasing sequence $\{\rho_n\}$, such that for any $n_1 \in N$, there exists an integer $n_2 > n_1$, with

$$\lim_{n\to\infty} \sup \sum_{s=0}^{m-1} \left(\frac{K\rho_s}{2^{\alpha-1}} \sum_{\xi=a}^{b} Q_{s,\xi} - \frac{1}{(\alpha+1)^{\alpha+1}} (1+p^\alpha) \frac{(\Delta\rho_s)^{\alpha+1}}{(\rho_s R_s)^\alpha} \right) = \infty, \tag{2.5}$$

where$Q_{n,\xi} = min\{q(n,\xi), (q(n,\xi) - \tau)\}$.

Then every solution of equation (1.1) is oscillatory.

Proof. Assume that $\{x_n\}$ is a positive solution of equation (1.1) which does not tend to zero as $n \to \infty$. From equation (1.1), we have

$$\Delta(a_n\psi(x_n)(\Delta z_n)^\alpha) \leq -\sum_{\xi=a}^{b} q(n,\xi)\, f\left(x\big(g(n,\xi)\big)\right) \leq 0. \tag{2.6}$$

From (2.6) and condition (A_4)there exists $n_2 \geq n_1$ such that for$n \geq n_2$, we get

$$0 = \Delta(a_n\psi(x_n)(\Delta z_n)^\alpha) + \sum_{\xi=a}^{b} q(n,\xi)\, f\left(x\big(g(n,\xi)\big)\right)$$

$$= \Delta(a_n\psi(x_n)(\Delta z_n)^\alpha) + \sum_{\xi=a}^{b} q(n,\xi)\, f\left(x\big(g(n,\xi)\big)\right)$$

$$+p^\alpha \left[\Delta(a_{n-\tau}\psi(x_{n-\tau})(\Delta z_{n-\tau})^\alpha) + \sum_{\xi=a}^{b} q\big((n,\xi)-\tau\big)\, f\big(x(g(n,\xi)-\tau)\big) \right]$$

$$= \Delta(a_n\psi(x_n)(\Delta z_n)^\alpha) + \sum_{\xi=a}^{b} q(n,\xi)\, f\left(x\big(g(n,\xi)\big)\right)$$

$$+p^\alpha[\Delta(a_{n-\tau}\psi(x_{n-\tau})(\Delta z_{n-\tau})^\alpha)] + p^\alpha \left[\sum_{\xi=a}^{b} q\big((n,\xi)-\tau\big)\, f\big(x(g(n,\xi)-\tau)\big) \right]$$

$$= \left[\Delta(a_n\psi(x_n)(\Delta z_n)^\alpha) + p^\alpha\big(\Delta(a_{n-\tau}\psi(x_{n-\tau})(\Delta z_{n-\tau})^\alpha)\big)\right]$$

$$+\sum_{\xi=a}^{b} \left\{ q(n,\xi) f\left(x\big(g(n,\xi)\big)\right) + p^\alpha q\big((n,\xi)-\tau\big) f\big(x(g(n,\xi)-\tau)\big) \right\}$$

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$$\geq \left[\Delta(a_n\psi(x_n)(\Delta z_n)^\alpha) + p^\alpha[\Delta(a_{n-\tau}\psi(x_{n-\tau})(\Delta z_{n-\tau})^\alpha)]\right]$$

$$+K\sum_{\xi=a}^{b} Q_{n,\xi}\{x^\alpha(g(n,\xi)) + p^\alpha x^\alpha(g(n,\xi)-\tau)\}.$$

By using Lemma 2.2, we obtain

$$\left[\Delta(a_n\psi(x_n)(\Delta z_n)^\alpha) + p^\alpha[\Delta(a_{n-\tau}\psi(x_{n-\tau})(\Delta z_{n-\tau})^\alpha)]\right] + \frac{K}{2^{\alpha-1}}\sum_{\xi=a}^{b} Q_{n,\xi} z^\alpha(g(n,\xi)) \leq 0.$$

Further, it is clear from (A_3)

$$g(n,\xi) \geq min\{g(n,a), g(n,b)\} \equiv G_n\,, \xi \in N(a,b).$$

Thus

$$\left[\Delta(a_n\psi(x_n)(\Delta z_n)^\alpha) + p^\alpha[\Delta(a_{n-\tau}\psi(x_{n-\tau})(\Delta z_{n-\tau})^\alpha)]\right] + \frac{K}{2^{\alpha-1}} z^\alpha(G_n)\sum_{\xi=a}^{b} Q_{n,\xi} \leq 0. \tag{2.7}$$

Define

$$\omega_n := \rho_n \frac{a_n\psi(x_n)(\Delta z_n)^\alpha}{z^\alpha(G_n)}. \tag{2.8}$$

Then$\omega_n > 0$. From (2.8), we have

$$\Delta\omega_n = \Delta\rho_n \frac{a_{n+1}\psi(x_{n+1})(\Delta z_{n+1})^\alpha}{z^\alpha(G_{n+1})} + \rho_n\Delta\left(\frac{a_n\psi(x_n)(\Delta z_n)^\alpha}{z^\alpha(G_n)}\right)$$

$$= \Delta\rho_n \frac{a_{n+1}\psi(x_{n+1})(\Delta z_{n+1})^\alpha}{z^\alpha(G_{n+1})} + \rho_n \frac{\Delta(a_n\psi(x_n)(\Delta z_n)^\alpha)}{z^\alpha(G_n)}$$

$$-\rho_n \frac{a_{n+1}\psi(x_{n+1})(\Delta z_{n+1})^\alpha \Delta(z^\alpha(G_n))}{z^\alpha(G_{n+1}) z^\alpha(G_n)}. \tag{2.9}$$

Since $\Delta z_n > 0$, and from the lemma (2.4), we have

$$\Delta(z^\alpha(G_n)) = z^\alpha(G_{n+1}) - z^\alpha(G_n) \geq \alpha z^{\alpha-1}(G_n)(z(G_{n+1}) - z(G_n))$$

$$= \alpha z^{\alpha-1}(G_n)\Delta(z(G_n))\ , \alpha \geq 1. \tag{2.10}$$

Substitute from (2.10)in(2.9), we have

$$\Delta\omega_n \leq \frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1} + \rho_n \frac{\Delta(a_n\psi(x_n)(\Delta z_n)^\alpha)}{z^\alpha(G_n)}$$

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$$-\alpha\rho_n \frac{a_{n+1}\psi(x_{n+1})(\Delta z_{n+1})^\alpha z^{\alpha-1}(G_n)\Delta\big(z(G_n)\big)}{z^{2\alpha}(G_{n+1})}. \tag{2.11}$$

By Lemma (2.1), since $a_n\psi(x_n)|\Delta z_n|^{\alpha-1}\Delta z_n = a_n\psi(x_n)(\Delta z_n)^\alpha$ is decreasing sequance then $a_n\psi(x_n)$ $(\Delta z_n)^\alpha \le a(G_n)\psi\big(x(G_n)\big)\big(\Delta z(G_n)\big)^\alpha$. Then it follows that

$$\frac{\Delta\big(z(G_n)\big)}{\Delta z_n} \ge \left(\frac{a_n\psi(x_n)}{a(G_n)\psi\big(x(G_n)\big)}\right)^{1/\alpha}. \tag{2.12}$$

It follows from (2.11)and (2.12) that

$$\Delta\omega_n \le \frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1} + \rho_n\frac{\Delta(a_n\psi(x_n)(\Delta z_n)^\alpha)}{z^\alpha(G_n)}$$

$$-\alpha\rho_n\frac{a_{n+1}\psi(x_{n+1})(\Delta z_{n+1})^\alpha}{z^\alpha(G_{n+1})}\frac{z^{\alpha-1}(G_n)}{z^\alpha(G_{n+1})}\frac{\big(a_n\psi(x_n)\big)^{1/\alpha}\Delta z_n}{\Big(a(G_n)\psi\big(x(G_n)\big)\Big)^{1/\alpha}}$$

$$= \rho_n\frac{\Delta(a_n\psi(x_n)(\Delta z_n)^\alpha)}{z^\alpha(G_n)} + \frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1} - \alpha\rho_n\frac{R_n}{\rho_{n+1}^{\alpha+1/\alpha}}\omega_{n+1}^{\frac{\alpha+1}{\alpha}}. \tag{2.13}$$

Similarly, define another sequence v_n by

$$v_n := \rho_n\frac{a_{n-\tau}\psi(x_{n-\tau})(\Delta z_{n-\tau})^\alpha}{z^\alpha(G_n)}. \tag{2.14}$$

Then $v_n > 0$ .From (2.14), we have

$$\Delta v_n = \frac{\Delta\rho_n}{\rho_{n+1}}v_{n+1} + \rho_n\Delta\left(\frac{a_{n-\tau}\psi(x_{n-\tau})(\Delta z_{n-\tau})^\alpha}{z^\alpha(G_n)}\right)$$

$$= \frac{\Delta\rho_n}{\rho_{n+1}}v_{n+1} + \rho_n\frac{\Delta(a_{n-\tau}\psi(x_{n-\tau})(\Delta z_{n-\tau})^\alpha)}{z^\alpha(G_n)}$$

$$-\rho_n\frac{a_{n+1-\tau}\psi(x_{n+1-\tau})(\Delta z_{n+1-\tau})^\alpha\Delta\big(z^\alpha(G_n)\big)}{z^\alpha(G_{n+1})z^\alpha(G_n)}. \tag{2.15}$$

From (2.14),(2.15)and (2.12), we have

$$\Delta v_n \le \rho_n\frac{\Delta(a_{n-\tau}\psi(x_{n-\tau})(\Delta z_{n-\tau})^\alpha)}{z^\alpha(G_n)} + \frac{\Delta\rho_n}{\rho_{n+1}}v_{n+1} - \alpha\rho_n\frac{R_n}{\rho_{n+1}^{\alpha+1/\alpha}}v_{n+1}^{\frac{\alpha+1}{\alpha}}. \tag{2.16}$$

From (2.13) and (2.16), we obtain

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$$\Delta\omega_n + p^\alpha \Delta v_n \le \rho_n \frac{[\Delta(a_n\psi(x_n)(\Delta z_n)^\alpha) + p^\alpha \Delta(a_{n-\tau}\psi(x_{n-\tau})(\Delta z_{n-\tau})^\alpha)]}{z^\alpha(G_n)}$$

$$+\frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1} - \alpha\rho_n \frac{R_n}{\rho_{n+1}^{\alpha+1/\alpha}}\omega_{n+1}^{\frac{\alpha+1}{\alpha}} + p^\alpha\left[\frac{\Delta\rho_n}{\rho_{n+1}}v_{n+1} - \alpha\rho_n\frac{R_n}{\rho_{n+1}^{\alpha+1/\alpha}}v_{n+1}^{\frac{\alpha+1}{\alpha}}\right]. \quad (2.17)$$

From (2.7) and (2.17), we have

$$\Delta\omega_n + p^\alpha\Delta v_n \le -\rho_n\frac{K}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{n,\xi} + \frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1} - \alpha\rho_n\frac{R_n}{\rho_{n+1}^{\alpha+1/\alpha}}\omega_{n+1}^{\frac{\alpha+1}{\alpha}}$$

$$+p^\alpha\left[\frac{\Delta\rho_n}{\rho_{n+1}}v_{n+1} - \alpha\rho_n\frac{R_n}{\rho_{n+1}^{\alpha+1/\alpha}}v_{n+1}^{\frac{\alpha+1}{\alpha}}\right]. \quad (2.18)$$

Using (2.18) and the inequality

$$Bu - Au^{\alpha+1/\alpha} \le \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^\alpha}, A > 0, \quad (2.19)$$

we have

$$\Delta\omega_n + p^\alpha\Delta v_n \le -\rho_n\frac{K}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{n,\xi} + \frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\Delta\rho_n)^{\alpha+1}}{(\rho_n R_n)^\alpha} + \frac{p^\alpha}{(\alpha+1)^{\alpha+1}}\frac{(\Delta\rho_n)^{\alpha+1}}{(\rho_n R_n)^\alpha}.$$

Summing the lastinequality from $n_2$to $n-1$, we obtain

$$\sum_{s=n_2}^{n-1}\left(\rho_s\frac{K}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{s,\xi} - \frac{1}{(\alpha+1)^{\alpha+1}}(1+p^\alpha)\frac{(\Delta\rho_s)^{\alpha+1}}{(\rho_s R_s)^\alpha}\right) \le \omega_{n_2} + p^\alpha v_{n_2}.$$

Which yields

$$\sum_{s=n_2}^{n-1}\left(\rho_s\frac{K}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{s,\xi} - \frac{1}{(\alpha+1)^{\alpha+1}}(1+p^\alpha)\frac{(\Delta\rho_s)^{\alpha+1}}{(\rho_s R_s)^\alpha}\right) \le c_1,$$

where$c_1 > 0$ is a finite constant. But, this contradicts(2.5). This completes the proof of Theorem 2.1. ■

Remark 2.1. Note that from Theorem 2.1, we can obtain different conditions for oscillation of all solutions of equation (1.1) by different choices of $\{\rho_n\}$. Let $\rho_n = n^\lambda, n \ge n_0$ and $\lambda > 1$ is a constant. By Theorem 2.1, we have the following result.

Corollary 2.1. Assume that all the assumptions of Theorem 2.1 hold, except the condition (2.5) is replaced by

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$$\limsup_{n\to\infty}\sum_{l=n_0}^{n}\left(\rho_l\frac{K}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{l,\xi}-(1+p^{\alpha})\frac{\left((l+1)^{\lambda}-l^{\lambda}\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}(l^{\lambda}R_l)^{\alpha}}\right)=\infty.$$

Then every solution of equation (1.1) is oscillatory.

Remark2.2. If $\psi(x_n)\equiv 1, \alpha\equiv 1, g(n,\xi)\equiv g(n), q(n,\xi)\equiv q(n)$. Then Theorem 2.1 extended and improved Theorem 1 in [19].

By using the inequality in Lemma 2.3, we obtain the following result.

Theorem 2.2. Let $0<\alpha\leq 1$. Further, assume that there exists a positive non decreasing sequence $\{\rho_n\}$, such that for any $n_1\in N$, there exists an integer $n_2>n_1$, with

$$\limsup_{m\to\infty}\sum_{n=n_0}^{m-1}\left(K\rho_n\sum_{\xi=a}^{b}Q_{n,\xi}-\frac{(1+p^{\alpha})}{(\alpha+1)^{\alpha+1}}\frac{(\Delta\rho_n)^{\alpha+1}}{(\rho_nR_n)^{\alpha}}\right)=\infty.$$

Then equation (1.1) is oscillatory.

Proof. The proof is similar to that of Theorem 2.1 and hence the details are omitted.

Theorem 2.3. Assume that$\alpha\geq 1$, and let $\{\rho_n\}$ be a positive sequence. Furthermore, we assume that there exists a double sequence $\{H_{m,n}|m\geq n\geq 0\}$. If

$$\limsup_{m\to\infty}\frac{1}{H_{m,0}}\sum_{n=0}^{m-1}\left(H_{m,n}\rho_n\frac{K}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{n,\xi}-(1+p^{\alpha})\frac{1}{(\alpha+1)^{\alpha+1}}\frac{\vartheta_{m,n}^{\alpha+1}H_{m,n}}{\Theta_n^{\alpha}}\right)=\infty. \qquad (2.20)$$

Then every solution of equation (1.1) is oscillatory.

Proof. Proceeding as in Theorem 2.1 we assume that equation (1.1) has a non- oscillatory solution, say $x_n>0$ and $x_{n-\tau}>0$ for all $n\geq n_0$. From the proofof Theorem 2.1, we find that (2.18) holds for all $n\geq n_2$. From (2.18), we have

$$\rho_n\frac{K}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{n,\xi}\leq-\Delta\omega_n-p^{\alpha}\Delta v_n+\frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1}-\alpha\Theta_n\omega_{n+1}^{\frac{\alpha+1}{\alpha}}+p^{\alpha}\left[\frac{\Delta\rho_n}{\rho_{n+1}}v_{n+1}-\alpha\Theta_nv_{n+1}^{\frac{\alpha+1}{\alpha}}\right].$$

Therefore, we have

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$$\sum_{n=k}^{m-1} H_{m,n}\rho_n \frac{K}{2^{\alpha-1}} \sum_{\xi=a}^{b} Q_{n,\xi} \leq -\sum_{n=k}^{m-1} H_{m,n}\,\Delta\omega_n - p^\alpha \sum_{n=k}^{m-1} H_{m,n}\,\Delta v_n + \sum_{n=k}^{m-1} H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1}$$

$$-\sum_{n=k}^{m-1} \alpha H_{m,n}\Theta_n\omega_{n+1}^{\frac{\alpha+1}{\alpha}} + p^\alpha \sum_{n=k}^{m-1} H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}} v_{n+1} - \alpha p^\alpha \sum_{n=k}^{m-1} H_{m,n}\Theta_n v_{n+1}^{\frac{\alpha+1}{\alpha}},$$

whichyields after summing by parts

$$\sum_{n=k}^{m-1} H_{m,n}\rho_n \frac{K}{2^{\alpha-1}} \sum_{\xi=a}^{b} Q_{n,\xi} = H_{m,k}\omega_k + \sum_{n=k}^{m-1}\left(\Delta_2 H_{m,n} + H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}}\right)\omega_{n+1} - \alpha\sum_{n=k}^{m-1} H_{m,n}\Theta_n\omega_{n+1}^{\alpha+1/\alpha}$$

$$+p^\alpha H_{m,k}v_k + p^\alpha \sum_{n=k}^{m-1}\left(\Delta_2 H_{m,n} + H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}}\right) v_{n+1} - \alpha p^\alpha \sum_{n=k}^{m-1} H_{m,n}\Theta_n v_{n+1}^{\alpha+1/\alpha}.$$

Hence

$$\sum_{n=k}^{m-1} H_{m,n}\rho_n \frac{K}{2^{\alpha-1}} \sum_{\xi=a}^{b} Q_{n,\xi} \leq H_{m,k}\omega_k + \sum_{n=k}^{m-1}\left(\frac{\Delta\rho_n}{\rho_{n+1}} - \frac{h_{m,n}}{\sqrt{H_{m,n}}}\right) H_{m,n}\omega_{n+1} - \alpha\sum_{n=k}^{m-1} H_{m,n}\Theta_n\omega_{n+1}^{\alpha+1/\alpha}$$

$$+p^\alpha H_{m,k}v_k + p^\alpha \sum_{n=k}^{m-1}\left(\frac{\Delta\rho_n}{\rho_{n+1}} - \frac{h_{m,n}}{\sqrt{H_{m,n}}}\right) H_{m,n}v_{n+1} - \alpha p^\alpha \sum_{n=k}^{m-1} H_{m,n}\Theta_n v_{n+1}^{\alpha+1/\alpha}$$

$$= H_{m,k}\omega_k + \sum_{n=k}^{m-1} \vartheta_{m,n} H_{m,n}\omega_{n+1} - \alpha\sum_{n=k}^{m-1} H_{m,n}\Theta_n\omega_{n+1}^{\alpha+1/\alpha}$$

$$+p^\alpha H_{m,k}v_k + p^\alpha \sum_{n=k}^{m-1} \vartheta_{m,n} H_{m,n}v_{n+1} - \alpha p^\alpha \sum_{n=k}^{m-1} H_{m,n}\Theta_n v_{n+1}^{\alpha+1/\alpha}.$$

From (2.19), we have

$$\sum_{n=k}^{m-1} H_{m,n}\rho_n \frac{K}{2^{\alpha-1}} \sum_{\xi=a}^{b} Q_{n,\xi} \leq H_{m,k}\omega_k$$

$$+\sum_{n=k}^{m-1} \frac{1}{(\alpha+1)^{\alpha+1}} \frac{\vartheta_{m,n}^{\alpha+1} H_{m,n}}{\Theta_n^\alpha} + p^\alpha H_{m,k}v_k + p^\alpha \sum_{n=k}^{m-1} \frac{1}{(\alpha+1)^{\alpha+1}} \frac{\vartheta_{m,n}^{\alpha+1} H_{m,n}}{\Theta_n^\alpha}.$$

Then,

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$$\sum_{n=k}^{m-1}\left(H_{m,n}\rho_n\frac{K}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{n,\xi}-(1+p^{\alpha})\frac{1}{(\alpha+1)^{\alpha+1}}\frac{\vartheta_{m,n}^{\alpha+1}H_{m,n}}{\Theta_n^{\alpha}}\right)\leq H_{m,k}\omega_k+p^{\alpha}H_{m,k}v_k,$$

which implies

$$\sum_{n=k}^{m-1}\left(H_{m,n}\rho_n\frac{K}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{n,\xi}-(1+p^{\alpha})\frac{1}{(\alpha+1)^{\alpha+1}}\frac{\vartheta_{m,n}^{\alpha+1}H_{m,n}}{\Theta_n^{\alpha}}\right)\leq H_{m,0}|\omega_k|+p^{\alpha}H_{m,0}|v_k|.$$

Hence,

$$\sum_{n=0}^{m-1}\left(H_{m,n}\rho_n\frac{K}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{n,\xi}-(1+p^{\alpha})\frac{1}{(\alpha+1)^{\alpha+1}}\frac{\vartheta_{m,n}^{\alpha+1}H_{m,n}}{\Theta_n^{\alpha}}\right)\leq$$

$$H_{m,0}\left\{\sum_{n=0}^{k-1}\left|\rho_n\frac{K}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{n,\xi}\right|+|\omega_k|+p^{\alpha}|v_k|\right\}.$$

Hence,

$$\lim_{m\to\infty}\sup\frac{1}{H_{m,0}}\sum_{n=0}^{m-1}\left(H_{m,n}\rho_n\frac{K}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{n,\xi}-(1+p^{\alpha})\frac{1}{(\alpha+1)^{\alpha+1}}\frac{\vartheta_{m,n}^{\alpha+1}H_{m,n}}{\Theta_n^{\alpha}}\right)$$

$$\leq\sum_{n=0}^{k-1}\left|\rho_n\frac{K}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{n,\xi}\right|+|\omega_k|+p^{\alpha}|v_k|<\infty,$$

which is contrary to (2.20). This completes the proof of Theorem 2.3. ■

Corollary 2.2. Assume that all the assumptions of Theorem 2.3 hold, except the condition (2.20) is replaced by

$$\lim_{m\to\infty}\sup\frac{1}{H_{m,0}}\sum_{n=0}^{m-1}H_{m,n}\rho_n\frac{K}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{n,\xi}=\infty,$$

$$\lim_{m\to\infty}\sup\frac{1}{H_{m,0}}\sum_{n=0}^{m-1}\frac{(1+p^{\alpha})}{(\alpha+1)^{\alpha+1}}\frac{\vartheta_{m,n}^{\alpha+1}H_{m,n}}{\Theta_n^{\alpha}}<\infty.$$

Then equation (1.1) is oscillatory.

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Remark2.3. By choosing specific sequence $H_{m,n}$, we can derive several oscillation criteria for (1.1) .Let us consider the double sequence $H_{m,n}$ defined by

$$H_{m,n} = (m-n)^{\lambda}, \lambda \geq 1, m \geq n \geq 0,$$

Then $H_{m,m} = 0$ for m ≥ 0 and $H_{m,n} > 0$ and $\Delta_2 H_{m,n} \leq 0$ for $m > n \geq 0$. By Theorem 2.3, we get the following oscillation criteria for (1.1).

Corollary2.3. Assume that all the assumptions of Theorem 2.2 hold, except the condition (2.20) is replaced by

$$\lim_{m\to\infty} \sup \frac{1}{m^{\lambda}} \sum_{n=0}^{m-1} \left((m-n)^{\lambda} \left(\rho_n \frac{K}{2^{\alpha-1}} \sum_{\xi=a}^{b} Q_{n,\xi} - \frac{(1+p^{\alpha})}{(\alpha+1)^{\alpha+1} (\rho_n R_n)^{\alpha}} \left(\Delta\rho_n - \frac{\lambda\rho_{n+1}}{(m-n)} \right)^{\alpha+1} \right) \right) = \infty.$$

Then equation (1.1) is oscillatory.

Remark 2.4. If $\psi(x_n) \equiv 1, p_n \equiv 0, g(n,\xi) \equiv n-\sigma, q(n,\xi) \equiv q(n)$then Theorem 2.1and 2.3 extended and improved Theorem 2.1and 2.2respectively in [15].

By using the inequality in Lemma 2.3, we obtain the following result.

Theorem2.4. Let $0 < \alpha \leq 1$. Further, assume that there exists a positive non decreasing sequence $\{\rho_n\}$, such that for any $n_1 \in N$, there existsan integer $n_2 > n_1$, with

$$\lim_{m\to\infty} \sup \frac{1}{H_{m,0}} \sum_{n=0}^{m-1} \left(H_{m,n}\rho_n K \sum_{\xi=a}^{b} Q_{n,\xi} - (1+p^{\alpha}) \frac{1}{(\alpha+1)^{\alpha+1}} \frac{\vartheta_{m,n}^{\alpha+1} H_{m,n}}{\Theta_n^{\alpha}} \right) = \infty.$$

Then equation (1.1) is oscillatory.

Proof. The proof is similar to that of Theorem 2.3 and hence the details are omitted.

Theorem 2.5. Let $\alpha \geq 1$. Further, assume that there exists a positive non decreasing sequence $\{\rho_n\}$, such that for any $n_1 \in N$, there exists an integer $n_2 > n_1$, with

$$\lim_{m\to\infty} \sup \sum_{l=n_0}^{m-1} \left(\frac{K\rho_l}{2^{\alpha-1}} \sum_{\xi=a}^{b} Q_{l,\xi} - \frac{(1+p^{\alpha})}{2^{3-\alpha}} \frac{(\Delta\rho_l)^2}{\rho_l J_1} \right) = \infty, \qquad (2.21)$$

where$J_n = R_n^{\alpha}$.

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Then every solution of equation (1.1) oscillatory.

Proof.Assume that $\{x_n\}$ is a positive solution of equation (1.1) which does not tend to zero as $n \to \infty$. By Lemma 2.1, we have (2.1) and from Theorem 2.1, we have (2.7).Define ω_n and v_n by (2.8) and (2.14) respectively. Proceeding as in the proof of Theorem 2.1, we obtain (2.9) and (2.15). By using the inequality
$x^\alpha - y^\alpha \geq 2^{1-\alpha}(x-y)^\alpha$ $for\ x \geq y > 0\ and\ \alpha \geq 1$,we have

$$\Delta\left(z^\alpha(G_n)\right) = z^\alpha(G_{n+1}) - z^\alpha(G_n) \geq 2^{1-\alpha}\left(z(G_{n+1}) - z(G_n)\right)^\alpha = 2^{1-\alpha}\left(\Delta z(G_n)\right)^\alpha \ ,\alpha \geq 1. \quad (2.22)$$

Substitute from (2.22)in (2.9), we have

$$\Delta\omega_n \leq \rho_n \frac{\Delta(a_n\psi(x_n)(\Delta z_n)^\alpha)}{z^\alpha(G_n)} + \frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1} - 2^{1-\alpha}\rho_n \frac{a_{n+1}\psi(x_{n+1})(\Delta z_{n+1})^\alpha\left(\Delta\left(z(G_n)\right)\right)^\alpha}{z^{2\alpha}(G_{n+1})}. \quad (2.23)$$

From (2.12), we have

$$\Delta\omega_n \leq \rho_n \frac{\Delta(a_n\psi(x_n)(\Delta z_n)^\alpha)}{z^\alpha(G_n)} + \frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1}$$

$$-2^{1-\alpha}\rho_n \frac{a_{n+1}\psi(x_{n+1})(\Delta z_{n+1})^\alpha}{z^\alpha(G_{n+1})} \frac{a_n\psi(x_n)(\Delta z_n)^\alpha}{z^\alpha(G_{n+1})} \frac{1}{a(G_n)\psi\left(x(G_n)\right)}$$

$$\leq \rho_n \frac{\Delta(a_n\psi(x_n)(\Delta z_n)^\alpha)}{z^\alpha(G_n)} + \frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1} - 2^{1-\alpha}\frac{\rho_n \mathrm{J_n}}{\rho_{n+1}^2}\omega_{n+1}^2. \quad (2.24)$$

On the other hand, from (2.15), we have

$$\Delta v_n \leq \rho_n \frac{\Delta(a_{n-\tau}\psi(x_{n-\tau})(\Delta z_{n-\tau})^\alpha)}{z^\alpha(G_n)} + \frac{\Delta\rho_n}{\rho_{n+1}} v_{n+1} - 2^{1-\alpha}\frac{\rho_n \mathrm{J_n}}{\rho_{n+1}^2} v_{n+1}^2. \quad (2.25)$$

From (2.24) and (2.25), we obtain

$$\Delta\omega_n + p^\alpha \Delta v_n \leq \rho_n \frac{[\Delta(a_n\psi(x_n)(\Delta z_n)^\alpha) + p^\alpha\Delta(a_{n-\tau}\psi(x_{n-\tau})(\Delta z_{n-\tau})^\alpha)]}{z^\alpha(G_n)}$$

$$+\frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1} - 2^{1-\alpha}\frac{\rho_n \mathrm{J_n}}{\rho_{n+1}^2}\omega_{n+1}^2 + p^\alpha\left[\frac{\Delta\rho_n}{\rho_{n+1}} v_{n+1} - 2^{1-\alpha}\frac{\rho_n \mathrm{J_n}}{\rho_{n+1}^2} v_{n+1}^2\right]. \quad (2.26)$$

From (2.7) and (2.26), we have

$$\Delta\omega_n + p^\alpha \Delta v_n \leq -\rho_n \frac{K}{2^{\alpha-1}}\sum_{\xi=a}^{b} Q_{n,\xi} + \frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1} - 2^{1-\alpha}\frac{\rho_n \mathrm{J_n}}{\rho_{n+1}^2}\omega_{n+1}^2$$

$$+p^\alpha\left[\frac{\Delta\rho_n}{\rho_{n+1}} v_{n+1} - 2^{1-\alpha}\frac{\rho_n \mathrm{J_n}}{\rho_{n+1}^2} v_{n+1}^2\right]. \quad (2.27)$$

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Using the inequality $Bu - Au^2 \leq \frac{B^2}{4A}$, $A > 0$ in (2.27), we have

$$\Delta\omega_n + p^\alpha \Delta v_n \leq -\rho_n \frac{K}{2^{\alpha-1}} \sum_{\xi=a}^{b} Q_{n,\xi} + \frac{1}{2^{3-\alpha}} (1+p^\alpha) \frac{(\Delta\rho_n)^2}{\rho_n \mathrm{J_n}}. \tag{2.28}$$

Summing (2.28)from $n_2$to $n-1$, we obtain

$$\sum_{s=n_2}^{n-1} \left(\rho_s \frac{K}{2^{\alpha-1}} \sum_{\xi=a}^{b} Q_{s,\xi} - \frac{1}{2^{3-\alpha}} (1+p^\alpha) \frac{(\Delta\rho_s)^2}{\rho_n \mathrm{J_s}} \right) \leq \omega_{n_2} + p^\alpha v_{n_2},$$

which yields

$$\sum_{s=n_2}^{n-1} \left(\rho_s \frac{K}{2^{\alpha-1}} \sum_{\xi=a}^{b} Q_{s,\xi} - \frac{(1+p^\alpha)}{2^{3-\alpha}} \frac{(\Delta\rho_s)^2}{\rho_s \mathrm{J_s}} \right) \leq c_1,$$

where$c_1 > 0$ is a finite constant. Taking lim sup in the above inequality, we obtain a contradiction with (2.21).

This completes the proof of Theorem 2.5. ■

By using the inequality in Lemma 2.3, we obtain the following result.

Theorem 2.6. Let $0 < \alpha \leq 1$. Further, assume that there exists a positive non decreasing sequence $\{\rho_n\}$, such that for any $n_1 \in N$, there exists an integer $n_2 > n_1$, with

$$\limsup_{n\to\infty} \sum_{n=n_2}^{n-1} \left(\rho_n K \sum_{\xi=a}^{b} Q_{n,\xi} - \frac{1}{2^{3-\alpha}} (1+p^\alpha) \frac{(\Delta\rho_n)^2}{\rho_n \mathrm{J_n}} \right) = \infty.$$

Then equation (1.1) is oscillatory.

Proof. The proof is similar to that of Theorem 2. 5 and hence the details are omitted.

Theorem2.7. Assume that $\alpha \geq 1$and let $\{\rho_n\}$ be a positive sequence. Furthermore, we assume that there exists a double sequence $\{H_{m,n} | m \geq n \geq 0\}$. If

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$$\lim_{m\to\infty}\sup\frac{1}{H_{m,0}}\sum_{n=0}^{m-1}\left(H_{m,n}\rho_n\frac{K}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{n,\xi}-(1+p^{\alpha})\frac{\vartheta_{m,n}^2H_{m,n}}{4\varphi_n}\right)=\infty. \quad (2.29)$$

Then every solution of equation (1.1) isoscillatory.

Proof.Proceeding as in Theorem 2.5 we assume that equation (1.1) has a non- oscillatory solution, say $x_n > 0$ and $x_{n-\tau} > 0$ for all $n \geq n_0$. From the proofof Theorem 2.5 we find that (2.27) holds for all $n \geq n_2$. From (2.27), we have

$$\rho_n\frac{K}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{n,\xi}\leq-\Delta\omega_n-p^{\alpha}\Delta v_n+\frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1}-\varphi_n\omega_{n+1}^2+p^{\alpha}\left[\frac{\Delta\rho_n}{\rho_{n+1}}v_{n+1}-\varphi_n v_{n+1}^2\right]. \quad (2.30)$$

Therefore, we have

$$\sum_{n=k}^{m-1}H_{m,n}\rho_n\frac{K}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{n,\xi}\leq-\sum_{n=k}^{m-1}H_{m,n}\,\Delta\omega_n-p^{\alpha}\sum_{n=k}^{m-1}H_{m,n}\,\Delta v_n$$

$$+\sum_{n=k}^{m-1}H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1}-\sum_{n=k}^{m-1}H_{m,n}\varphi_n\omega_{n+1}^2$$

$$+p^{\alpha}\sum_{n=k}^{m-1}H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}}v_{n+1}-p^{\alpha}\sum_{n=k}^{m-1}H_{m,n}\varphi_n v_{n+1}^2,$$

Which yields after summing by parts

$$\sum_{n=k}^{m-1}H_{m,n}\rho_n\frac{K}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{n,\xi}\leq H_{m,k}\omega_k+\sum_{n=k}^{m-1}\left(\Delta_2H_{m,n}+H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}}\right)\omega_{n+1}-\sum_{n=k}^{m-1}H_{m,n}\varphi_n\omega_{n+1}^2$$

$$+p^{\alpha}H_{m,k}v_k+p^{\alpha}\sum_{n=k}^{m-1}\left(\Delta_2H_{m,n}+H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}}\right)v_{n+1}-p^{\alpha}\sum_{n=k}^{m-1}H_{m,n}\varphi_n v_{n+1}^2$$

$$=H_{m,k}\omega_k+\sum_{n=k}^{m-1}\left(\frac{\Delta\rho_n}{\rho_{n+1}}-\frac{h_{m,n}}{\sqrt{H_{m,n}}}\right)H_{m,n}\omega_{n+1}-\sum_{n=k}^{m-1}H_{m,n}\varphi_n\omega_{n+1}^2$$

$$+p^{\alpha}H_{m,k}v_k+p^{\alpha}\sum_{n=k}^{m-1}\left(\frac{\Delta\rho_n}{\rho_{n+1}}-\frac{h_{m,n}}{\sqrt{H_{m,n}}}\right)H_{m,n}v_{n+1}-p^{\alpha}\sum_{n=k}^{m-1}H_{m,n}\varphi_n v_{n+1}^2$$

$$=H_{m,k}\omega_k+\sum_{n=k}^{m-1}\vartheta_{m,n}H_{m,n}\omega_{n+1}-\sum_{n=k}^{m-1}H_{m,n}\varphi_n\omega_{n+1}^2$$

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$$+p^{\alpha}H_{m,k}v_k+p^{\alpha}\sum_{n=k}^{m-1}\vartheta_{m,n}H_{m,n}v_{n+1}-p^{\alpha}\sum_{n=k}^{m-1}H_{m,n}\varphi_n v_{n+1}^2.$$

Using the inequality $Bu - Au^2 \le \frac{B^2}{4A}, A > 0$, we have

$$\sum_{n=k}^{m-1}H_{m,n}\rho_n\frac{K}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{n,\xi}\le H_{m,k}\omega_k+\sum_{n=k}^{m-1}\frac{\vartheta_{m,n}^2H_{m,n}}{4\varphi_n}-\sum_{n=k}^{m-1}\left[\sqrt{H_{m,n}\varphi_n}\omega_{n+1}+\frac{\vartheta_{m,n}}{2}\sqrt{\frac{H_{m,n}}{\varphi_n}}\right]^2$$

$$+p^{\alpha}H_{m,k}v_k+p^{\alpha}\sum_{n=k}^{m-1}\frac{\vartheta_{m,n}^2H_{m,n}}{4\varphi_n}-\sum_{n=k}^{m-1}\left[\sqrt{H_{m,n}\varphi_n}v_{n+1}+\frac{\vartheta_{m,n}}{2}\sqrt{\frac{H_{m,n}}{\varphi_n}}\right]^2. \quad (2.31)$$

Then,

$$\sum_{n=k}^{m-1}\left(H_{m,n}\rho_n\frac{K}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{n,\xi}-(1+p^{\alpha})\frac{\vartheta_{m,n}^2H_{m,n}}{4\varphi_n}\right)\le H_{m,k}\omega_k+p^{\alpha}H_{m,k}v_k,$$

which implies

$$\sum_{n=k}^{m-1}\left(H_{m,n}\rho_n\frac{K}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{n,\xi}-(1+p^{\alpha})\frac{\vartheta_{m,n}^2H_{m,n}}{4\varphi_n}\right)\le H_{m,0}|\omega_k|+p^{\alpha}H_{m,0}|v_k|.$$

Hence,

$$\sum_{n=0}^{m-1}\left(H_{m,n}\rho_n\frac{K}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{n,\xi}-(1+p^{\alpha})\frac{\vartheta_{m,n}^2H_{m,n}}{4\varphi_n}\right)\le H_{m,0}\left\{\sum_{n=0}^{k-1}\left|\rho_n\frac{K}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{n,\xi}\right|+|\omega_k|+p^{\alpha}|v_k|\right\}.$$

Hence,

$$\lim_{m\to\infty}\sup\frac{1}{H_{m,0}}\sum_{n=0}^{m-1}\left(H_{m,n}\rho_n\frac{K}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{n,\xi}-(1+p^{\alpha})\frac{\vartheta_{m,n}^2H_{m,n}}{4\varphi_n}\right)$$

$$\le\sum_{n=0}^{k-1}\left|\rho_n\frac{K}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{n,\xi}\right|+|\omega_k|+p^{\alpha}|v_k|<\infty,$$

which is contrary to (2.29). This completes the proof of Theorem 2.7.■

Corollary 2.4. Assume that all the assumptions of Theorem 2.7 hold, except the condition (2.29) is replaced by

$$\lim_{m\to\infty}\sup\frac{1}{H_{m,0}}\sum_{n=0}^{m-1}H_{m,n}\rho_n\frac{K}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{n,\xi}=\infty,$$

$$\lim_{m\to\infty}\sup\frac{1}{H_{m,0}}\sum_{n=0}^{m-1}(1+p^{\alpha})\frac{\vartheta_{m,n}^2H_{m,n}}{4\varphi_n}<\infty.$$

Then equation (1.1) is oscillatory.

Remark2.5. By choosing specific sequence $H_{m,n}$, we can derive severaloscillation criteria for (1.1).Let us consider the double sequence $H_{m,n}$ defined by

$$H_{m,n}=(m-n)^{\lambda},\lambda\geq 1,m\geq n\geq 0.$$

By Theorem 2.7, we get the following oscillation criteria for (1.1).

Corollary2.5. Assume that all the assumptions of Theorem 2.7 hold, except the condition (2.29) is replaced by

$$\lim_{m\to\infty}\sup\frac{1}{m^{\lambda}}\sum_{n=0}^{m-1}\left((m-n)^{\lambda}\left(\frac{K}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{n,\xi}-(1+p^{\alpha})\frac{\left(\frac{\Delta\rho_n}{\rho_{n+1}}-\frac{\lambda}{(m-n)}\right)^2}{4\varphi_n}\right)\right)=\infty.$$

Then equation (1.1) is oscillatory.

Remark2.6. If $\psi(x_n)\equiv 1, g(n,\xi)\equiv n^{\alpha}, q(n,\xi)\equiv q(n)$. Then we reduced to Theorems of Sakerin [14].

Remark2.7. If$\psi(x_n)\equiv 1,\alpha\equiv 1, g(n,\xi)\equiv n+1-l,\ q(n,\xi)\equiv q(n)$.Then we reduced to Theorems in [18].

Remark2.8. If$\psi(x_n)\equiv 1,\alpha\equiv 1, g(n,\xi)\equiv n-\tau, q(n,\xi)\equiv q(n)$.Then Theorem2.7 extended and improved Theorem 1 in [12].

By using the inequality in Lemma 2.3, we obtain the following result.

Theorem2.8. Let $0<\alpha\leq 1$. Further, assume that there exists a positive non decreasing sequence $\{\rho_n\}$, such that for any $n_1\in N$, there exists an integer $n_2>n_1$, with

$$\lim_{m\to\infty}\sup\frac{1}{H_{m,0}}\sum_{n=0}^{m-1}\left(KH_{m,n}\rho_s\sum_{\xi=a}^{b}Q_{s,\xi}-(1+p^{\alpha})\frac{\vartheta_{m,n}^2H_{m,n}}{4\varphi_n}\right)=\infty.$$

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Then equation (1.1) is oscillatory.

Proof. The proof is similar to that of Theorem 2.7 and hence the details are omitted.

Theorem 2.9.Assume that there exists a positive non decreasing sequence $\{\rho_n\}$, such that for any $n_1 \in N$, there exists an integer $n_2 > n_1$, with

$$\lim_{m\to\infty} \sup \sum_{n=0}^{m-1} \left(H_{m,n}\Psi_n - \mu_{m,n}\right) = \infty, \tag{2.32}$$

where

$$\delta_{n,\xi} := \sum_{\xi=a}^{b} q(n,\xi)\left(1 - p\left(g(n,\xi)\right)\right)^{\alpha}.$$

Then every solution of equation (1.1) is oscillatory.

Proof. Assume that $\{x_n\}$ is a positive solution of equation (1.1) which does not tend to zero as $n \to \infty$.

From (2.1) and the fact that $x_n \le z_n$, we see that

$$x(g(n,\xi) - \tau) \le z(g(n,\xi) - \tau) \le z\left(g(n,\xi)\right), n \in N(n_2), \xi \in N(a,b) \tag{2.33}$$

Further, it is clear form (A_3) that

$$g(n,\xi) \ge min\{g(n,a), g(n,b)\} \equiv G_n , \xi \in N(a,b).$$

Which in view of (2.1) leads to

$$z\left(g(n,\xi)\right) \ge z(G_n), n \in N(n_3) , \xi \in N(a,b) for some n_3 \ge n_2.$$

Using the above inequality together with(2.1),(2.33),$(\mathrm{A}_3) and (\mathrm{A}_4)$ in equation (1.1)for $n \ge n_3$, we get

$$\begin{aligned} 0 &= \Delta(a_n\psi(x_n)(\Delta z_n)^{\alpha}) + \sum_{\xi=a}^{b} q(n,\xi)\, f\left(x\left(g(n,\xi)\right)\right) \\ &\ge \Delta(a_n\psi(x_n)(\Delta z_n)^{\alpha}) + K\sum_{\xi=a}^{b} q(n,\xi)\left|x\left(g(n,\xi)\right)\right|^{\alpha} \\ &= \Delta(a_n\psi(x_n)(\Delta z_n)^{\alpha}) + K\sum_{\xi=a}^{b} q(n,\xi)\left(z\left(g(n,\xi)\right) - p\left(g(n,\xi)\right)x(g(n,\xi) - \tau)\right)^{\alpha} \end{aligned}$$

$$\geq \Delta(a_n\psi(x_n)(\Delta z_n)^\alpha) + K\sum_{\xi=a}^{b} q(n,\xi)\left(1-p\big(g(n,\xi)\big)\right)^\alpha z^\alpha\big(g(n,\xi)\big)$$

$$\geq \Delta(a_n\psi(x_n)(\Delta z_n)^\alpha) + K\sum_{\xi=a}^{b} q(n,\xi)\left(1-p\big(g(n,\xi)\big)\right)^\alpha z^\alpha(G_n)$$

$$= \Delta(a_n\psi(x_n)(\Delta z_n)^\alpha) + Kz^\alpha(G_n)\sum_{\xi=a}^{b} q(n,\xi)\left(1-p\big(g(n,\xi)\big)\right)^\alpha$$

$$= \Delta(a_n\psi(x_n)(\Delta z_n)^\alpha) + Kz^\alpha(G_n)\delta_{n,\xi}. \quad (2.34)$$

Define the sequence ω_n by the generalized Riccati substitution

$$\omega_n \coloneqq \rho_n a_n \psi(x_n)\left[\frac{(\Delta z_n)^\alpha}{z^\alpha(G_n)} + \beta_n\right] .n \geq n_3 \quad (2.35)$$

It follows that

$$\Delta\omega_n = \Delta(\rho_n a_n\psi(x_n)\beta_n) + a_{n+1}\psi(x_{n+1})(\Delta z_{n+1})^\alpha \Delta\left[\frac{\rho_n}{z^\alpha(G_n)}\right] + \frac{\rho_n\Delta(a_n\psi(x_n)(\Delta z_n)^\alpha)}{z^\alpha(G_n)}.$$

From (2.34) and(2.35), we have

$$\Delta\omega_n \leq -K\rho_n\delta_{n,\xi} + \rho_n\Delta(a_n\psi(x_n)\beta_n) + \frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1} - \rho_n\frac{a_{n+1}\psi(x_{n+1})(\Delta z_{n+1})^\alpha\Delta\big(z^\alpha(G_n)\big)}{z^\alpha(G_{n+1})z^\alpha(G_n)}. \quad (2.36)$$

First: we consider the case when $\alpha \geq 1$. By using the inequality([5, see $P.39$])

$$x^\alpha - y^\alpha \geq \alpha y^{\alpha-1}(x-y) \; for \; x \neq y > 0 \; and \; \alpha \geq 1,$$

We may write

$$\Delta\big(z^\alpha(G_n)\big) = z^\alpha(G_{n+1}) - z^\alpha(G_n) \geq \alpha z^{\alpha-1}(G_n)\big(z(G_{n+1}) - z(G_n)\big) = \alpha z^{\alpha-1}(G_n)\Delta\big(z(G_n)\big) \; ,\alpha \geq 1.$$

Substituting in(2.36), we have

$$\Delta\omega_n \leq -K\rho_n\delta_{n,\xi} + \rho_n\Delta(a_n\psi(x_n)\beta_n) + \frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1}$$

$$-\alpha\rho_n\frac{a_{n+1}\psi(x_{n+1})(\Delta z_{n+1})^\alpha z^{\alpha-1}(G_n)\Delta\big(z(G_n)\big)}{z^{2\alpha}(G_{n+1})}. \quad (2.37)$$

From (2.12)and (2.37), we find

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$$\Delta\omega_n \le -K\rho_n\delta_{n,\xi} + \rho_n\Delta(a_n\psi(x_n)\beta_n) + \frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1}$$

$$-\alpha\rho_n \frac{a_{n+1}\psi(x_{n+1})(\Delta z_{n+1})^\alpha}{z^\alpha(G_{n+1})}\frac{z^{\alpha-1}(G_n)}{z^\alpha(G_{n+1})}\frac{\left(a_n\psi(x_n)\right)^{1/\alpha}\Delta z_n}{\left(a(G_n)\psi\left(x(G_n)\right)\right)^{1/\alpha}}$$

$$= -K\rho_n\delta_{n,\xi} + \rho_n\Delta(a_n\psi(x_n)\beta_n) + \frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1} - \alpha\rho_n R_n\left(\frac{a_{n+1}\psi(x_{n+1})(\Delta z_{n+1})^\alpha}{z^\alpha(G_{n+1})}\right)^{\frac{\alpha+1}{\alpha}}. \quad (2.38)$$

Second: we consider the case when $0 < \alpha < 1$. By using the inequality

$$x^\alpha - y^\alpha \ge \alpha x^{\alpha-1}(x-y)\ for\ x \ne y > 0\ and\ 0 < \alpha < 1,$$

We may write

$$\Delta\left(z^\alpha(G_n)\right) = z^\alpha(G_{n+1}) - z^\alpha(G_n) \ge \alpha z^{\alpha-1}(G_{n+1})\left(z(G_{n+1}) - z(G_n)\right) = \alpha z^{\alpha-1}(G_{n+1})\Delta\left(z(G_n)\right)\ , 0 < \alpha < 1.$$

Substituting in(2.36), we have

$$\Delta\omega_n \le -K\rho_n\delta_{n,\xi} + \rho_n\Delta(a_n\psi(x_n)\beta_n) + \frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1} - \alpha\rho_n\frac{a_{n+1}\psi(x_{n+1})(\Delta z_{n+1})^\alpha z^{\alpha-1}(G_{n+1})\Delta\left(z(G_n)\right)}{z^{2\alpha}(G_{n+1})}.$$

From (2.12) and by Lemma(2.1), since$a_n\psi(x_n)(\Delta z_n)^\alpha$is decreasing sequence, we have

$$-\alpha\rho_n\frac{a_{n+1}\psi(x_{n+1})(\Delta z_{n+1})^\alpha z^{\alpha-1}(G_{n+1})\Delta\left(z(G_n)\right)}{z^{2\alpha}(G_{n+1})} \le -\alpha\rho_n R_n\left(\frac{a_{n+1}\psi(x_{n+1})(\Delta z_{n+1})^\alpha}{z^\alpha(G_{n+1})}\right)^{\frac{\alpha+1}{\alpha}}. \quad (2.39)$$

Thus, we again obtain (2.38). However, from (2.35) we see that

$$\left(\frac{a_{n+1}\psi(x_{n+1})(\Delta z_{n+1})^\alpha}{z^\alpha(G_{n+1})}\right)^{\frac{\alpha+1}{\alpha}} = \left(\frac{\omega_{n+1}}{\rho_{n+1}} - a_{n+1}\psi(x_{n+1})\beta_{n+1}\right)^{1+\frac{1}{\alpha}}. \quad (2.40)$$

Then, by using the inequality([7, see p. 534])

$$(v-u)^{1+\frac{1}{\alpha}} \ge v^{1+\frac{1}{\alpha}} + \frac{1}{\alpha}u^{1+\frac{1}{\alpha}} - \left(1+\frac{1}{\alpha}\right)u^{\frac{1}{\alpha}}v\,, \qquad \alpha = \frac{odd}{odd} \ge 1,$$

we may write equation (2.40) as follows

$$\left(\frac{\omega_{n+1}}{\rho_{n+1}} - a_{n+1}\psi(x_{n+1})\beta_{n+1}\right)^{1+\frac{1}{\alpha}} \ge \left(\frac{\omega_{n+1}}{\rho_{n+1}}\right)^{1+\frac{1}{\alpha}} + \frac{(a_{n+1}\psi(x_{n+1})\beta_{n+1})^{1+\frac{1}{\alpha}}}{\alpha}$$

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$$-\frac{\left(1+\frac{1}{\alpha}\right)(a_{n+1}\psi(x_{n+1})\beta_{n+1})^{\frac{1}{\alpha}}}{\rho_{n+1}}\omega_{n+1}.$$

Substituting back in(2.38), we have

$$\Delta\omega_n \le -K\rho_n\delta_{n,\xi} + \rho_n\Delta(a_n\psi(x_n)\beta_n) - \rho_nR_n(a_{n+1}\psi(x_{n+1})\beta_{n+1})^{1+\frac{1}{\alpha}}$$

$$+\left(\Delta\rho_n + \alpha\rho_nR_n\left(1+\frac{1}{\alpha}\right)(a_{n+1}\psi(x_{n+1})\beta_{n+1})^{\frac{1}{\alpha}}\right)\frac{\omega_{n+1}}{\rho_{n+1}} - \left(\frac{\alpha\rho_nR_n}{\rho_{n+1}^{1+\frac{1}{\alpha}}}\right)\omega_{n+1}^{1+\frac{1}{\alpha}}. \qquad (2.41)$$

Thus,

$$\Psi_n \le -\Delta\omega_n + \frac{\eta_n}{\rho_{n+1}}\omega_{n+1} - \left(\frac{\alpha\rho_nR_n}{\rho_{n+1}^{1+\frac{1}{\alpha}}}\right)\omega_{n+1}^{1+\frac{1}{\alpha}}. \quad n \ge k$$

Therefore, we have

$$\sum_{n=k}^{m-1} H_{m,n}\Psi_n \le -\sum_{n=k}^{m-1} H_{m,n}\,\Delta\omega_n + \sum_{n=k}^{m-1}\frac{\eta_nH_{m,n}}{\rho_{n+1}}\omega_{n+1} - \sum_{n=k}^{m-1}\left(H_{m,n}\frac{\alpha\rho_nR_n}{\rho_{n+1}^{1+\frac{1}{\alpha}}}\right)\omega_{n+1}^{1+\frac{1}{\alpha}},$$

which yields after summing by parts

$$\sum_{n=k}^{m-1} H_{m,n}\Psi_n \le H_{m,k}\omega_k + \sum_{n=k}^{m-1}\Delta_2H_{m,n}\omega_{n+1} + \sum_{n=k}^{m-1}\frac{\eta_nH_{m,n}}{\rho_{n+1}}\omega_{n+1} - \sum_{n=k}^{m-1}\left(H_{m,n}\frac{\alpha\rho_nR_n}{\rho_{n+1}^{1+\frac{1}{\alpha}}}\right)\omega_{n+1}^{1+\frac{1}{\alpha}}.$$

Hence

$$\sum_{n=k}^{m-1} H_{m,n}\Psi_n \le H_{m,k}\omega_k + \sum_{n=k}^{m-1}\left(\frac{\eta_nH_{m,n}}{\rho_{n+1}} - h_{m,n}\sqrt{H_{m,n}}\right)\omega_{n+1} - \sum_{n=k}^{m-1}\left(\frac{\alpha\rho_nR_nH_{m,n}}{\rho_{n+1}^{1+\frac{1}{\alpha}}}\right)\omega_{n+1}^{1+\frac{1}{\alpha}}.$$

Using the inequality

$$Bu - Au^{1+\frac{1}{\alpha}} \le \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}\frac{B^{1+\alpha}}{A^\alpha},$$

for$A = \frac{\alpha\rho_nR_nH_{m,n}}{\rho_{n+1}^{1+\frac{1}{\alpha}}}$ $and\ B = \left(\frac{\eta_nH_{m,n}}{\rho_{n+1}} - h_{m,n}\sqrt{H_{m,n}}\right)$, we obtain

$$\sum_{n=k}^{m-1}\left(H_{m,n}\Psi_n - \mu_{m,k}\right) \le H_{m,k}\omega_k \le H_{m,0}|\omega_k|,$$

which implies

$$\sum_{n=0}^{m-1}\left(H_{m,n}\Psi_n - \mu_{m,k}\right) \le H_{m,0}\left\{\sum_{n=0}^{k-1}\Psi_n + |\omega_k|\right\}.$$

Hence,

$$\lim_{m\to\infty}\sup\frac{1}{H_{m,0}}\sum_{n=0}^{m-1}\left(H_{m,n}\Psi_n - \mu_{m,k}\right) < \infty,$$

which is contrary to (2.32). This completes the proof of Theorem 2.9. ■

Corollary 2.6. Assume that all the assumptions of Theorem 2.9 hold, except the condition (2.32) is replaced by

$$\lim_{m\to\infty}\sup\frac{1}{H_{m,0}}\sum_{n=0}^{m-1}H_{m,n}\Psi_n = \infty,$$

$$\lim_{m\to\infty}\sup\frac{1}{H_{m,0}}\sum_{n=0}^{m-1}\mu_{m,k} < \infty.$$

Then equation (1.1) is oscillatory.

Examples

In order to show the application of our results obtained in this paper, let us consider the following second order difference equation with distributed deviating arguments:

Example3.1. Consider the nonlinear delay difference equation

$$\Delta\left(\frac{n}{n+1}\Delta x_n\right) + \sum_{\xi=0}^{1}\frac{\lambda\xi}{n^2}x_n(\xi + x_n^2) = 0, n \ge 1 \quad (3.1)$$

where$a_n = \frac{n}{n+1}, \psi(x_n) = 1, p_n = 0, \alpha = 1, q(n,\xi) = \frac{\lambda\xi}{n^2}$.If we take$\rho_n = n, K = 1$ then we have $R_l = \frac{n+1}{n}$,

$$\sum_{l=n_0}^{n}\left(\frac{K\rho_l}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{l,\xi} - \frac{(1+p^\alpha)\left((l+1)-l\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}(lR_l)^\alpha}\right)$$

$$= \sum_{l=1}^{n}\left(\frac{\lambda l}{l^2} - \frac{l}{4l(l+1)}\right) = \sum_{l=1}^{n}\left(\frac{\lambda}{l} - \frac{1}{4(l+1)}\right) \geq \sum_{l=1}^{n}\left(\frac{4\lambda - 1}{4l}\right) \to \infty$$

$as\ n \to \infty$ if $\lambda > \frac{1}{4}$. Thus Theorem 2.1 asserts that every solution of (3.1) is oscillatory when $\lambda > \frac{1}{4}$.

Example 3.2. Consider the linear neutral delay difference equation

$$\Delta^2(x_n + px_{n-1}) + \sum_{\xi=0}^{1}\frac{\lambda\xi}{n^2}x_{n+\xi} = 0, n \geq 1 \qquad (3.2)$$

where $a_n = \psi(x_n) = 1, p_n = p > 0, \alpha = 1, \tau = 1, q(n,\xi) = \frac{\lambda\xi}{n^2}$. If we take $\rho_n = n, K = 1 = n$, then, we have $R_l = 1$,

$$\sum_{l=n_0}^{n}\left(\frac{K\rho_l}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{l,\xi} - \frac{(1+p^\alpha)\big((l+1)-l\big)^{\alpha+1}}{(\alpha+1)^{\alpha+1}(lR_l)^\alpha}\right) = \sum_{l=1}^{n}\left(\frac{\lambda l}{l^2} - \frac{(1+p)}{4l}\right) = \sum_{l=1}^{n}\left(\frac{4\lambda - (1+p)}{4l}\right) = \infty$$

If $\lambda > (1+p)$. By Corollary 2.1, every solution of (3.2) oscillatory when $\lambda > (1+p)$.

Example 3.3. Consider the nonlinear delay difference equation

$$\Delta(n^2\Delta x_n) + \sum_{\xi=0}^{1}\lambda\xi\, x_n = 0, \qquad n \geq 1 \qquad (3.3)$$

where $a_n = n^2, \psi(x_n) = 1, p_n = 0. \alpha = 1, q(n,\xi) = \lambda\xi$. If we take $\rho_n = n, K = 1$, then we have $R_l = \frac{1}{l^2}$,

$$\sum_{l=n_0}^{n}\left(K\rho_l\sum_{\xi=a}^{b}Q_{l,\xi} - \frac{\big((l+1)-l\big)^2}{(\alpha+1)^{\alpha+1}(lR_l)^\alpha}\right) = \sum_{l=1}^{n}\left(\lambda l - \frac{l^2}{4l}\right) = \sum_{l=1}^{n}\frac{(4\lambda-1)l}{4} \to \infty$$

$as\ n \to \infty\ if\ \lambda > \frac{1}{4}$. By Theorem 2.2 every solution of (3.3) is oscillatory when $\lambda > \frac{1}{4}$.

Example 3.4. Consider the nonlinear neutral difference equation

$$\Delta\left(\frac{1}{n^3}\big(\Delta(x_n + px_{n-1})\big)^3\right) + \sum_{\xi=0}^{1}\frac{\lambda\xi}{n^3}x_{n-1}^3 = 0, \qquad n \geq 1 \qquad (3.4)$$

where $a_n = \frac{1}{n^3}, \psi(x_n) = 1, p_n = p > 0, \alpha = 3, q(n,\xi) = \frac{\lambda\xi}{n^3}$.If we take $\rho_n = n^2, K = 1$, then, we have $J_l = l^3$,

$$\lim_{n\to\infty} \sup \sum_{l=n_0}^{n-1} \left(\frac{K\rho_l}{2^{\alpha-1}} \sum_{\xi=a}^{b} Q_{l,\xi} - \frac{(1+p^\alpha)}{2^{3-\alpha}} \frac{(\Delta\rho_l)^2}{\rho_l J_l} \right) = \lim_{n\to\infty} \sup \sum_{l=1}^{n-1} \left(\frac{\lambda}{4l} - \frac{4(1+p^3)}{(l-1)^3} \right) = \infty$$

$if\ \lambda > 0$.By Theorem 2.5 every solution of (3.4) is oscillatory when $\lambda > 0$.

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