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APPLICATIONS OF DIFFERENTIAL SUBORDINATION TO CERTAIN SUBCLASSES OF MEROMORPHICALLY UNIVALENT FUNCTIONS WITH DIFFERENTIAL OPERATOR

ABSTRACT

By making use of the principle of differential subordination, we investigate several subordination and convolution properties of certain subclasses of meromorphic univalent functions which are defined here by means of a differential operator. We also indicate relevant connections of the various results presented in this paper with those obtained in earlier works.

2010 Mathematics Subject Classification: Primary **30C45**.

Keywords: Meromorphic functions, Differential subordination, Hadamard product (or convolution), Differential operator, Hypergeometric function.

1. Introduction and Definitions

Let Σ_m denote the class of functions of the form :

$$f(z)=\frac{1}{z}+\sum_{k=m}^{\infty}a_k z^k \qquad (m\in N := \{1,2,3,.....\}), \tag{1.1}$$

which are analytic in the punctured unit disc $U^*=\{z\ :\ z\in C \ \text{ and }\ 0<|z|<1\}$, with a simple pole at the origin.

If $f(z)$ and $g(z)$ are analytic in U, we say that $f(z)$ is subordinate to $g(z)$, written symbolically as follows:

$$f\prec g \ \text{ in } U \quad \text{or} \quad f(z)\prec g(z) \quad (z\in U),$$

if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U, with

$$w(0)=0 \ \text{ and } \ |w(z)|<1 \quad (z\in U)$$

such that

$$f(z)=g(w(z)) \qquad\qquad (z\in U).$$

If $g(z)$ is univalent in U, we have the following equivalence relationship holds true: (cf., e.g., [5]; see also [6,p.4]):

$$f(z)\prec g(z) \quad (z\in U)\Leftrightarrow f(0)=g(0) \ \text{ and } \ f(U)\subset g(U).$$

For functions $f(z)\in\Sigma_m$, given by (1.1), and $g(z)\in\Sigma_m$ defined by

$$g(z)=\frac{1}{z}+\sum_{k=m}^{\infty}b_k z^k, \tag{1.2}$$

we define the Hadamard product (or convolution) of $f(z)$ and $g(z)$ by

$$(f*g)(z):=\frac{1}{z}+\sum_{k=m}^{\infty}a_k b_k z^k =: (g*f)(z) \qquad (z\in U). \tag{1.3}$$

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In a recent paper, Frasin [1] defined the following differential operator:

$$I_\lambda^0 f(z) = f(z)$$

$$I_\lambda^1 f(z) = (1-\lambda) f(z) + \lambda z f'(z) + \frac{2\lambda}{z}, \quad \lambda \geq 0,$$

$$I_\lambda^2 f(z) = (1-\lambda) I_\lambda^1 f(z) + \lambda z (I_\lambda^1 f(z)' + \frac{2\lambda}{z},$$

and for $n = 1,2,3,...$

$$I_\lambda^n f(z) = (1-\lambda) I_\lambda^{n-1} f(z) + \lambda z (I_\lambda^{n-1} f(z))' + \frac{2\lambda}{z}$$

$$= \frac{1}{z} + \sum_{k=m}^{\infty} [1+\lambda(k-1)]^n a_k z^k. \qquad (1.4)$$

Note that for $\lambda = m = 1,$ we have the operator $I^n f(z)$ introduced and studied by Frasin and Darus [2].

It easily verified from (1.4) that

$$\lambda z (I_\lambda^n f(z))' = I_\lambda^{n+1} f(z) - (1-\lambda) I_\lambda^n f(z) - \frac{2\lambda}{z}. \qquad (1.5)$$

Definition. For fixed parameters A and B $(-1 \leq B < A \leq 1),$ we say that a function $f(z) \in \Sigma_m,$ is in the class $\Sigma_m^n(\lambda, A, B),$ if it satisfies the following subordination condition:

$$-z^2 (I_\lambda^n f(z))' \prec \frac{1+Az}{1+Bz} \qquad (n \in N_0;\ z \in U). \qquad (1.6)$$

In view of the definition of differential subordination, (1.6) is equivalent to the following condition:

$$\left| \frac{z^2 (I_\lambda^n f(z))' + 1}{B z^2 (I_\lambda^n f(z))' + A} \right| < 1 \quad (z \in U).$$

For convenience, we write

$$\Sigma_m^n (1-2\alpha, -1) = \Sigma_m^n (\lambda, \alpha),$$

where $\Sigma_m^n(\lambda, \alpha)$ denotes the class of functions in Σ_m satisfying the following inequality

$$R(-z^2 (I_\lambda^n f(z))') > \alpha \quad (0 \leq \alpha < 1;\ z \in U).$$

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In this paper, we derive several subordination and convolution properties for the function class $\Sigma_m^n(\lambda, A, B)$, which we have defined here by means of the differential operator $I_\lambda^n f$. Relevant connections of the various results presented in this paper with those obtained in earlier works are also pointed out.

2. Preliminary Lemmas

In proving our main results, we need each of the following lemmas.

Lemma 1 (Miller and Mocanu [5]; see also [6]).

Let the function $h(z)$ be analytic and convex (univalent) in U with $h(0)=1$. Suppose also that the function $\phi(z)$ given by

$$\phi(z) = 1 + c_{m+1}z^{m+1} + c_{m+2}z^{m+2} + \ldots, \qquad (2.1)$$

is analytic in U. If

$$\phi(z) + \frac{z\phi'(z)}{\gamma} \prec h(z) \quad (R(\gamma) \geq 0; \gamma \neq 0; z \in U), \qquad (2.2)$$

then

$$\phi(z) \prec \psi(z) = \frac{\gamma}{m+1} z^{-\frac{\gamma}{m+1}} \int_0^z t^{\frac{\gamma}{m+1}-1} h(t)dt \prec h(z) \quad (z \in U),$$

and $\psi(z)$ is the best dominant of (2.2).

with a view to stating a well-known result (Lemma 2 below), we denote by $P(\gamma)$ the class of functions $\varphi(z)$ given by

$$\varphi(z) = 1 + b_1 z + b_2 z^2 + \ldots\ldots, \qquad (2.3)$$

which are analytic in U and satisfy the following inequality:

$$R(\varphi(z)) > \gamma \ (0 \leq \gamma < 1; \ z \in U).$$

Lemma 2 (cf., e.g., Pashkouleva [7]). Let the function $\varphi(z)$, given by (2.3), be in the class $P(\gamma)$. Then

$$R\varphi(z)\} \geq 2\gamma - 1 + \frac{2(1-\gamma)}{1+|z|} \quad (0 \leq \gamma < 1; z \in U).$$

Lemma 3 (see [10]). For $0 \leq \gamma_1, \gamma_2 < 1$,

$$P(\gamma_1) * P(\gamma_2) \subset P(\gamma_3) \quad (\gamma_3 := 1 - 2(1-\gamma_1)(1-\gamma_2)).$$

The result is the beast possible.

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For real or complex numbers a, b and c $(c \notin z_0^- := \{0,-1,-2,...\})$, the Gauss hypergeometric function is defined by

$$ {}_2F_1(a,b;c;z) = 1 + \frac{ab}{c}.\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}.\frac{z^2}{2!} + ...$$

We note that the above series converges absolutely for $z \in U$ and hence represents an analytic function in U (see, for details, [11, chapter 14]).

Each of the identities (asserted by Lemma 4 below) is well-known (cf., e.g., [11, chapter 14]).

Lemma 4. For real or complex parameters a, b and c $(c \notin z_0^-)$,

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt$$

$$= \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\,{}_2F_1(a,b;c;z)(R(c)) > R(b) > 0); \qquad (2.4)$$

$$ {}_2F_1(a,b;c;z) = (1-z)^{-a}\,{}_2F_1(a,c-b;c;\frac{z}{z-1}); \qquad (2.5)$$

$$ {}_2F_1(a,b;c;z) = {}_2F_1(a,b-1;c;z) + \frac{az}{c}\,{}_2F_1(a+1,b;c+1;z); \quad (2.6)$$

$$ {}_2F_1(a,b;\frac{a+b+1}{2};\frac{1}{2}) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{a+b+1}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}. \qquad (2.7)$$

3. The Main Subordination Theorems and The Associated Functional Inequalities

Unless otherwise mentioned, we shell assume throughout the sequel that

$$m \in N, \quad -1 \le B < A \le 1, \; \lambda > 0, \quad \text{and} \;\; n \in N_0 = N \cup \{0\}.$$

Theorem 1.

Let the function $f(z)$ defined by (1.1) satisfy the following subordination condition:

$$-\frac{(1-\delta)z^2(I_\lambda^n f(z))' + \delta z^2 (I_\lambda^{n+1} f(z))'}{1-2\lambda\delta} - \frac{2\lambda\delta}{1-2\lambda\delta} \prec \frac{1+Az}{1+Bz}(z \in U).$$

Then

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$$-z^2(I_\lambda^n f(z))' \prec Q(z) \prec \frac{1+Az}{1+Bz}(z \in U), \tag{3.1}$$

where the function $Q(z)$ given by

$$Q(z)=\begin{cases} \frac{A}{B}+(1-\frac{A}{B})(1+Bz)^{-1}{}_2F_1(1,1;\frac{1-2\lambda\delta}{\lambda\delta(m+1)}+1;\frac{Bz}{1+Bz}) & (B\neq 0) \\ 1+\frac{(1-2\lambda\delta)AZ}{\lambda\delta(m-1)+1} & (B=0) \end{cases}$$

is the best dominant of (3.1). Furthermore,

$$R(-z^2(I_\lambda^n f(z))' > \rho \qquad (z \in U), \tag{3.2}$$

where

$$\rho=\begin{cases} \frac{A}{B}+(1-\frac{A}{B})(1+B)^{-1}{}_2F_1(1,1;\frac{1-2\lambda\delta}{\lambda\delta(m+1)}+1;\frac{B}{1-B}) & (B\neq 0) \\ 1-\frac{(1-2\lambda\delta)A}{\lambda\delta(m-1)+1} & (B=0). \end{cases}$$

The inequality in (3.2) is the best possible.

Proof. Consider the function $\phi(z)$ defined by

$$\phi(z)=-z^2(I_\lambda^n f(z))' \qquad (z\in U). \tag{3.3}$$

Then $\phi(z)$ is of the form (2.1) and is analytic in U. Applying the identity (1.5) in (3.3) and differentiating the resulting equation with respect to z, we get

$$-\frac{(1-\delta)z^2(I_\lambda^n f(z))'+\delta z^2(I_\lambda^{n+1} f(z))'}{1-2\lambda\delta}-\frac{2\lambda\delta}{1-2\lambda\delta}$$
$$=\phi(z)+\frac{\lambda\delta z\phi'(z)}{1-2\lambda\delta}\prec\frac{1+Az}{1+Bz}(z\in U).$$

Now, by using Lemma 1 for $\gamma=\frac{1-2\lambda\delta}{\lambda\delta}$, we deduce that

$$-z^2(I_\lambda^n f(z))' \prec Q(z)=\frac{1-2\lambda\delta}{\lambda\delta(m+1)}Z^{-\frac{1-2\lambda\delta}{\lambda\delta(m+1)}}\int_0^z t^{\frac{1-2\lambda\delta}{\lambda\delta(m+1)}-1}\left(\frac{1+At}{1+Bt}\right)dt$$

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$$= \begin{cases} \frac{A}{B} + (1 - \frac{A}{B})(1 + Bz)^{-1} {}_2F_1(1,1; \frac{1-2\lambda\delta}{\lambda\delta(m+1)} + 1; \frac{Bz}{1+Bz}) & (B \neq 0) \\ 1 + \frac{(1-2\lambda\delta)AZ}{\lambda\delta(m-1)+1} & (B = 0), \end{cases}$$

by change of variables followed by the use of the identities (2.4), (2.5) and (2.6) (with $b = 1$ and $c = a + 1$).This prove the assertion (3.1) of Theorem 1.

Next, in order to prove the assertion (3.2) of Theorem 1, it suffices to show that

$$\inf_{|z|<1} \{R(Q(z))\} = Q(-1). \qquad (3.4)$$

Indeed, for $|z| \leq r < 1$.

$$R\left(\frac{1+Az}{1+Bz}\right) \geq \frac{1-Ar}{1-Br} \quad (|z| \leq r < 1).$$

Upon setting

$$G(s,z) = \frac{1+Asz}{1+Bsz} \text{ and}$$

$$dv(s) = \frac{1-2\lambda\delta}{\lambda\delta(m+1)} s^{\frac{1-2\lambda\delta}{\lambda\delta(m+1)}-1} ds (0 \leq s \leq 1),$$

which is a positive measure on the closed interval [0,1], we get

$$Q(z) = \int_0^1 G(s,z) dv(s),$$

so that

$$R(Q(z)) \geq \int_0^1 \left(\frac{1-Asr}{1-Bsr}\right) dv(s) = Q(-r) \quad (|z| \leq r < 1).$$

Letting $r \to 1-$ in the above inequality, we obtain the assertion (3.2) of Theorem 1.

Finally, the estimate in (3.2) is the best possible as the function $Q(z)$ is the best dominant of (3.1).

Putting $\delta = 1$ in Theorem 1, we get the following result

Corollary 1.

If $f(z) \in \Sigma_m$ satisfies

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$$\frac{-z^2\left\{(I_\lambda^n f(z))' + \lambda z(I_\lambda^n f(z))''\right\}}{1-2\lambda} \prec \frac{1+Az}{1+Bz} \qquad (z \in U)$$

then

$$-z^2(I_\lambda^n f(z))' \prec Q(z) \prec \frac{1+Az}{1+Bz} \qquad (z \in U)$$

where the function $Q(z)$ is given by

$$Q(z) = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1+Bz)^{-1}{}_2F_1(1,1;\frac{1-2\lambda}{\lambda(m+1)}+1;\frac{Bz}{1+Bz}) & (B \neq 0) \\ 1+\frac{(1-2\lambda)AZ}{\lambda(m-1)+1} & (B=0) \end{cases}$$

is the best dominant. Furthermore

$$R(-z^2(I_\lambda^n f(z))') > p \qquad (z \in U),$$

where

$$\rho = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1+B)^{-1}{}_2F_1(1,1;\frac{1-2\lambda}{\lambda(m+1)}+1;\frac{B}{1-B}) & (B \neq 0) \\ 1-\frac{(1-2\lambda)A}{\lambda(m-1)+1} & (B=0). \end{cases}$$

The result is the beast possible.

Remark.

The result (asserted by corollary 1 above) when $m=0$ was also obtained by Patel and Sahoo[9] and Lashin 3].

For $A = 1-2\gamma(0 \le \gamma < 1),\ B=-1,\ m=0$ and $\lambda = \frac{1}{3}$

Corollary 1 yields the following result which obtained by Lashin [3]

Corollary 2.

If $f(z) \in \Sigma_m$ satisfies the following inequality

$$-z^2[3f'(z) + zf''(z)] > \gamma \qquad (0 \le \gamma < 1, z \in U)$$

the $\operatorname{Re}\{-z^2 f'(z)\} > 1+2(1-\gamma)(\ell n2-1)(z \in U)$

the result is best possible.

Theorem 2.

If $f(z) \in \Sigma_m^n(\alpha) \quad (0 \le \alpha < 1)$, then

$$R\left(\frac{-z^2\left\{(1-\delta)(I_\lambda^n f(z))' + \delta(I_\lambda^{n+1} f(z))' + 2\lambda\delta(1-\alpha)\right\}}{1-2\lambda\delta} \right) > \alpha \qquad (|z| < R_1), \quad (3.5)$$

where

$$R_1 = \left(\sqrt{1 + \frac{\lambda^2\delta^2(m+1)^2}{(1-2\lambda\delta)^2}} - \frac{\lambda\delta(m+1)}{1-2\lambda\delta} \right)^{\frac{1}{m+1}}.$$

The result is the best possible.

Proof. We begin by writing

$$-z^2(I_\lambda^n f(z))' = \alpha + (1-\alpha)u(z) \qquad (z \in U). \qquad (3.6)$$

Then, clearly, $u(z)$ is of the form (2.1), is analytic in U, and has a positive real part in U. Making use of the identity (1.5) in (3.6) and differentiating the resulting equation with respect to z. we observe that

$$-\frac{z^2[(1-\delta)(I_\lambda^n f(z))' + \delta(I_\lambda^{n+1} f(z))'] + 2\lambda\delta(1-\alpha) + \alpha}{[1-2\lambda\delta](1-\alpha)} = u(z) + \frac{\lambda\delta z u'(z)}{1-2\lambda\delta}. \qquad (3.7)$$

Now, by applying the following estimate [5]:

$$\frac{|zu'(z)|}{R\{u(z)\}} \le \frac{2(m+1)r^{m+1}}{1-r^{2(m+1)}} \qquad (|z| = r < 1),$$

in (3.7), we get

$$R\left(-\frac{z^2[(1-\delta)(I_\lambda^n f(z))' + \delta(I_\lambda^{n+1} f(z))'] + 2\lambda\delta(1-\alpha) + \alpha}{[1-2\lambda\delta](1-\alpha)} \right)$$

$$\ge R(u(z)).\left(1 - \frac{2\lambda\delta(m+1)r^{m+1}}{(1-2\lambda\delta)(1-r^{2(m+1)})} \right). \qquad (3.8)$$

It is easily seen that the right- hand side of (3.8) is positive, provided that $r < R_1$, where R_1 is given as in Theorem 2. This proves the assertion (3.5) of Theorem 2.

In order to show that the bound R_1 is the best possible, we consider the function $f(z) \in \Sigma_m$ defined by

Noting that

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$$\frac{-z^2\left[(1-\delta)(I_\lambda^n f(z))' + \delta(I_\lambda^{n+1} f(z))'\right] + 2\lambda\delta(1-\alpha)+\alpha}{[1-2\lambda\delta](1-\alpha)}$$

$$= \frac{(1-2\lambda\delta)(1-z^{2(m+1)}) + 2\lambda\delta(m+1)z^{m+1}}{(1-2\lambda\delta)(1-z^{m+1})^2} = 0$$

for

$$z = R.\exp\left(\frac{i\pi}{m+1}\right).$$

we complete the proof of Theorem 2.

Theorem 3.

Let $f(z) \in \Sigma_m^n(\lambda, A, B)$ and let

$$F_\gamma(f)(z) = \frac{\gamma}{z^{\gamma+1}} \int_0^z t^\gamma f(t)dt \qquad (\gamma > 0; z \in U^*). \tag{3.9}$$

Then

$$-z^2(I_\lambda^n F_\gamma(f)(z))' \prec \theta(z) \prec \frac{1+Az}{1+Bz} (z \in U), \tag{3.10}$$

where the function $\theta(z)$ given by

$$\theta(z) = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1+Bz)^{-1}{}_2F_1(1,1;\frac{\gamma}{m+1}+1;\frac{Bz}{1+Bz}) & (B \neq 0) \\ 1 + \frac{\gamma Az}{\gamma+m+1} & (B=0). \end{cases}$$

is the best dominant of (3.10). Furthermore,

$$R\left(-z^2(I_\lambda^n F_\gamma(f)(z))'\right) > \chi \qquad (z \in U), \tag{3.11}$$

where

$$\chi = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1-Bz)^{-1}{}_2F_1(1,1;\frac{\gamma}{m+1}+1;\frac{B}{B-1}) & (B \neq 0) \\ 1 - \frac{\gamma Az}{\gamma+m+1} & (B=0). \end{cases}$$

The result is the best possible

Proof. Setting

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$$\phi(z) = -z^2 (I_\lambda^n F_\delta(f)(z))' \qquad (z \in U), \tag{3.12}$$

we note that $\phi(z)$ is of the form (2.1) and is analytic in U. Using the following operator identity:

$$z(I_\lambda^n F_\gamma(f)(z))' = \gamma I_\lambda^n f(z) - (\gamma+1) I_\lambda^n F_\gamma(f)(z), \tag{3.13}$$

in (3.12), and differentiating the resulting equation with respect to z, we fined that

$$-z^2(I_\lambda^n f(z))' = \phi(z) + \frac{z\phi'(z)}{\gamma} \prec \frac{1+Az}{1+Bz} \quad (z \in U).$$

Now the remaining part of Theorem 3 follows by employing the techniques that we used in proving Theorem 1 above.

Theorem 4.

Let $f(z) \in \Sigma_m$ *. Suppose also that* $g(z) \in \Sigma_m$ satisfies the following inequality:

$$R(zI_\lambda^n g(z)) > 0 \qquad (z \in U).$$

$$\left| \frac{I_\lambda^n f(z)}{I_\lambda^n g(z)} - 1 \right| < 1 \qquad (m \in N_0; z \in U),$$

then

$$R\left(-\frac{z(I_\lambda^n f(z))'}{I_\lambda^n f(z)} \right) > 0 \qquad (|z| < R_0),$$

where

$$R_0 = \frac{\sqrt{(m+2)+(m+1)^2} - (m+1)}{(m+2)}.$$

Proof. Letting

$$w(z) = \frac{I_\lambda^n f(z)}{I_\lambda^n g(z)} - 1 = k_{m+1} z^{m+1} + k_{m+2} z^{m+2} + \dots, \tag{3.14}$$

we note that $w(z)$ is analytic in U, with

$$w(0) = 0 \text{ and } |w(z)| \le |z|^{m+1} \quad (z \in U).$$

Then, by applying the familiar Schwarz Lemma, we get

$$w(z) = z^{m+1} \psi(z),$$

where the function $\psi(z)$ is analytic in U and

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$$|\psi(z)| \leq 1 \qquad (z \in U).$$

Therefore, (3.14) leads us to

$$I_\lambda^n f(z) = I_\lambda^n g(z)(1 + z^{m+1}\psi(z)) \qquad (z \in U). \tag{3.15}$$

Making use of logarithmic differentiation in (3.15), we obtain

$$\frac{z(I_\lambda^n f(z))'}{I_\lambda^n f(z)} = \frac{z(I_\lambda^n g(z))'}{I_\lambda^n g(z)} + \frac{z^{m+1}[(m+1)\psi(z) + z\psi'(z)]}{1 + z^{m+1}\psi(z)}. \tag{3.16}$$

Setting $\phi(z) = zI_\lambda^n g(z)$, we see that the function $\phi(z)$ is of the form (2.1), is analytic

in U,

$$R(\phi(z)) > 0 \qquad (z \in U),$$

and

$$\frac{z(I_\lambda^n g(z))'}{I_\lambda^n g(z)} = \frac{z\phi'(z)}{\phi(z)} - 1,$$

so that we fined from (3.16) that

$$R\left(-\frac{z(I_\lambda^n f(z))'}{I_\lambda^n f(z)}\right) \geq 1 - \left|\frac{z\phi'(z)}{\phi(z)}\right| - \left|\frac{z^{m+1}\{(m+1)\psi(z) + z\psi'(z)\}}{1 + z^{m+1}\psi(z)}\right| \qquad (z \in U). \tag{3.17}$$

Now, by using the following known estimates [8] (see also [4]):

$$\left|\frac{\phi'(z)}{\phi(z)}\right| \leq \frac{2(m+1)r^m}{1 - r^{2(m+1)}}$$

and

$$\left|\frac{(m+1)\psi(z) + z\psi'(z)}{1 + z^{m+1}\psi(z)}\right| \leq \frac{(m+1)}{1 - r^{m+1}} \qquad (|z| = r < 1)$$

in (3.18), we obtain

$$R\left(\frac{-z(I_\lambda^n f(z))'}{I_\lambda^n f(z)}\right) \geq \frac{1 - 2(m+1)r^{m+1} - (m+2)r^{2(m+1)}}{1 - r^{2(m+1)}} (|z| = r < 1),$$

which is certainly positive, provided that $r < R_0$, R_0 being given as in Theorem 4.

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Theorem 5.

Let $-1 \le B_j < A_j \le 1 \quad (j=1,2)$. *If each of the functions* $f_j(z) \in \Sigma_m$ satisfies the following subordination condition:

$$\frac{(1-\delta)zI_\lambda^n f_j(z)+\delta zI_\lambda^{n+1} f_j(z)}{1-2\lambda\delta}-\frac{2\lambda\delta}{1-2\lambda\delta} \prec \frac{1+A_j z}{1+B_j z} \quad (j=1,2;\ z\in U), \qquad (3.18)$$

then

$$\frac{(1-\delta)zI_\lambda^n H(z)+\delta zI_\lambda^{n+1} H(z)}{1-2\lambda\delta}-\frac{2\lambda\delta}{1-2\lambda\delta} \prec \frac{1+(1-2\eta)z}{1-z}(z\in U) \qquad (3.19)$$

where

$$H(z)=I_\lambda^n(f_1 * f_2)(z)$$

and

$$\eta = 1-\frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left[1-\frac{1}{2}\ {}_2F_1(1,1;\frac{1}{\lambda\delta}-1;\frac{1}{2})\right].$$

The result is the best possible when $B_1 = B_2 = -1$.

Proof. Suppose that each of the functions $f_j(z)\in\Sigma_p \quad (j=1,2)$ satisfies the condition (3.18). Then, by letting

$$\varphi_j(z)=\frac{(1-\delta)zI_\lambda^n f_j(z)+\delta zI_\lambda^{n+1} f_j(z)}{1-2\lambda\delta}-\frac{2\lambda\delta}{1-2\lambda\delta} \quad (j=1,2), \qquad (3.20)$$

we have

$$\varphi_j(z)\in p(\gamma_j) \qquad \left(\gamma_j=\frac{1-A_j}{1-B_j}, \qquad j=1,2\right).$$

By making use of the operator identity (1.5) in (3.20), we observe that

$$I_\lambda^n f_j(z)=\left(\frac{1}{\lambda\delta}-2\right)z^{1-\frac{1}{\lambda\delta}}\int_0^z t^{\frac{1}{\lambda\delta}-3} \quad \varphi_j(t)dt \qquad (j=1,2),$$

which, in view of the definition of $H(z)$ given already with (3.19), yields

$$I_\lambda^n H(z)=\left(\frac{1}{\lambda\delta}-2\right)z^{1-\frac{1}{\lambda\delta}}\int_0^z t^{\frac{1}{\lambda\delta}-3} \quad \varphi_0(t)dt, \qquad (3.21)$$

where, for convenience,

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$$\varphi_0(z) = \frac{(1-\delta)zI_\lambda^n H(z) + \delta zI_\lambda^{n+1} H(z)}{1-2\lambda\delta} - \frac{2\lambda\delta}{1-2\lambda\delta}$$

$$= \left(\frac{1}{\lambda\delta} - 2\right) z^{1-\frac{1}{\lambda\delta}} \int_0^z t^{\frac{1}{\lambda\delta}-3} \quad (\varphi_1 * \varphi_2)(t)dt. \qquad (3.22)$$

Since $\varphi_1(z) \in p(\gamma_1)$ and $\varphi_2(z) \in p(\gamma_2)$, it follows from Lemma 3 that

$$(\varphi_1 * \varphi_2)(z) \in p(\gamma_3) \qquad (\gamma_3 = 1 - 2(1-\gamma_1)(1-\gamma_2)). \qquad (3.23)$$

Now, by using (3.23) in (3.22) and then appealing to Lemma 2 and Lemma 4,we get

$$R\{\varphi_0(z)\} = \left(\frac{1}{\lambda\delta} - 2\right) \int_0^1 u^{\frac{1}{\lambda\delta}-3} R\{(\varphi_1 * \varphi_2)\}(uz)du.$$

$$\geq \left(\frac{1}{\lambda\delta} - 2\right) \int_0^1 u^{\frac{1}{\lambda\delta}-3} \left(2\gamma_3 - 1 + \frac{2(1-\gamma_3)}{1+u|z|}\right) du$$

$$> \left(\frac{1}{\lambda\delta} - 2\right) \int_0^1 u^{\frac{1}{\lambda\delta}-3} \left(2\gamma_3 - 1 + \frac{2(1-\gamma_3)}{1+u}\right) du$$

$$= 1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)} \left(1 - \left(\frac{1}{\lambda\delta} - 2\right) \int_0^1 u^{\frac{1}{\lambda\delta}-3} (1+u)^{-1} du\right)$$

$$= 1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)} \left[1 - \frac{1}{2}\, {}_2F_1\left(1,1;\left(\frac{1}{\lambda\delta} - 1\right);\frac{1}{2}\right)\right]$$

$$= \eta \qquad (z \in U).$$

when B_1 ■ B_2 ■ 1, we consider the functions $f_j(z) \in \Sigma_m$ $(j = 1, 2)$, which satisfy the hypothesis (3.18) of Theorem 5 and are defined by

$$I_\lambda^n f_j(z) = \left(\frac{1}{\lambda\delta} - 2\right) z^{1-\frac{1}{\lambda\delta}} \int_0^z t^{\frac{1}{\lambda\delta}-3} \left(\frac{1+A_j t}{1-t}\right) \quad dt (j = 1, 2).$$

Thus it follows from (3.22) and Lemma 4 that

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$$\varphi_0(z)=\left(\frac{1}{\lambda\delta}-2\right)\int_0^1 u^{\frac{1}{\lambda\delta}-3}\left(1-(1+A_1)(1+A_2)+\frac{(1+A_1)(1+A_2)}{1-uz}\right)du$$
$$=1-(1+A_1)(1+A_2)+(1+A_1)(1+A_2)(1-z)^{-1}{}_2F_1\left(1,1;\frac{1}{\lambda\delta}-1;\frac{z}{z-1}\right)$$
$$\to 1-(1+A_1)(1+A_2)+\frac{1}{2}(1+A_1)(1+A_2){}_2F_1\left(1,1;\frac{1}{\lambda\delta}-1;\frac{1}{2}\right) \text{ as } z\to -1,$$

which evidently completes the proof of Theorem 5.

Theorem 6.

If $f(z)\in\Sigma_m$ satisfies the following subordination condition:

$$\frac{(1-\delta)zI_\lambda^n f(z)+\delta zI_\lambda^{n+1}f(z)}{1-2\lambda\delta}-\frac{2\lambda\delta}{1-2\lambda\delta}\prec\frac{1+Az}{1+Bz}\quad(z\in U),$$

then

$$R\left((zI_\lambda^n f(z))^{1/q}\right)>\rho^{1/q}\quad(q\in N;z\in U),$$

where ρ is given as in Theorem 1. The result is the best possible

Proof. Defining the function $\phi(z)$ by

$$\phi(z)=zI_\lambda^n f(z)\quad(f\in\Sigma_m;z\in U),\tag{3.24}$$

we see that the function $\phi(z)$ is of the form (2.1) and is analytic in U. Using the identity (1.5) in (3.24) and differentiating the resulting equation with respect to z, we find that

$$\frac{(1-\delta)zI_\lambda^n f(z)+\delta zI_\lambda^{n+1}f(z)}{1-2\lambda\delta}-\frac{2\lambda\delta}{1-2\lambda\delta}=\phi(z)+\frac{\lambda\delta z\phi'(z)}{1-2\lambda\delta}\prec\frac{1+Az}{1+Bz}\quad(z\in U).$$

Now, by following the lines of proof of Theorem 1, and using the elementary inequality:

$$R(w^{1/q})\geq(R(w))^{1/q}\qquad(R(w))>0;q\in N),$$

we arrive at the result asserted by Theorem 6.

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