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# LATTICE POINTS ON THE HOMOGENEOUS CONE $X^2 + Y^2 = 40Z^2$

**ABSTRACT**

The ternary quadratic homogeneous equation representing homogeneous cone given by $X^2 + Y^2 = 40Z^2$ is analyzed for its non-zero distinct integer points on it . Seven different patterns of integer points satisfying the cone under consideration are obtained. A few interesting relation between the solutions and special number patterns namely Polygonal number , Pyramidal number , Octahedral number and Nasty number are presented. Also knowing an integer solution satisfying the given cone , two triples of integers generated from the given solution are exhibited.

**Keywords:** Ternary homogeneous quadratic ,integral solutions

**2010 Mathematics Subject Classification:11D09**

## 1. INTRODUCTION

The ternary quadratic Diophantine equations offer an unlimited field for research due to their variety[1,20]. For an extensive review of various problems, one may refer[2-19]. This communication concerns with yet another interesting ternary quadratic equation $X^2 + Y^2 = 40Z^2$ representing a cone for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

## NOTATIONS

$P_n^m$ - Pyramidal umber of rank n with size m.

$T_{m,n}$ -Polygonal number of rank n with size m.

$\Pr_n$ - Pronic number of rank n

$OH_n$ - Octahedral number of rank n

## 2. METHOD OF ANALYSIS

The ternary quadratic equation to be solved for its non-integer solutions is

$$X^2 + Y^2 = 40Z^2 \tag{1}$$

Assume $Z = Z(a,b) = a^2 + b^2; a,b > 0$ (2)

We illustrate below seven different patterns of non-zero distinct integer solutions to (1)

### 2.1 PATTERN: 1

Write 40 as

$$40 = (6+2i)(6-2i) \tag{3}$$

Substituting (2) and (3) in (1) and employing the method of factorization, define

$$(X+iY)(X-iY) = (6+2i)(6-2i)(a+ib)^2(a-ib)^2$$

Equating real and imaginary parts, we have

$$X = X(a,b) = 6a^2 - 6b^2 - 4ab$$

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$$Y = Y(a,b) = 2a^2 - 2b^2 + 12ab \qquad (4)$$

Thus (4) and (2) represent non-zero distinct integral solutions of (1) in two parameters.

**Properties:**

- $20^2 Z^2(A,B) - [3X(A,B) + Y(A,B)]^2$ is a perfect square
- $X(A,1) - T_{14,A} - A = -6$
- $Y(A,1) - T_{6,A} \equiv -2 \pmod{13}$
- $Y(3B,2) - T_{38,B} \equiv -8 \pmod{89}$
- $X(A+5, A+5) + T_{10,A} \equiv -14 \pmod{43}$
- $X[A, A(A+1)] - 6T_{4,A} + 8P_A^5 = 6 * 4T_{3,A}^2$ a nasty number

**2.2 PATTERN: 2**

Instead of (3),write 40 as

$$40 = (2+6i)(2-6i) \qquad (5)$$

Following the procedure presented in pattern:1 ,the corresponding values of X and Y obtained from (1) are

$$\left.\begin{aligned} X &= X(a,b) = 2a^2 - 2b^2 - 12ab \\ Y &= Y(a,b) = 6a^2 - 6b^2 + 4ab \end{aligned}\right\} \qquad (6)$$

Thus (2) and (6) represents non-zero distinct integer solutions of (1) in two parameters.

**Properties**

- $20^2 Z^2(A,B) - [X(A,B) + 3Y(A,B)]^2$ is a perfect square
- $T_{20,A} - X(A,1) - Y(A,1) - Z(A,1) \equiv 0 \pmod 7$
- $X(A,1) - T_{6,A} \equiv -2 \pmod{11}$
- $40Z(A,1) - X(A,1) - 2T_{40,A} \equiv 42 \pmod{48}$
- $Y(A+3, A+5) - T_{10,A} \equiv -3 \pmod{11}$
- $6Y(B,B), 3X(A,-4A)$ represent nasty number respectively

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**2.3 PATTERN: 3**

The ternary quadratic equation (1) can be written as

$$X^2 - 4Z^2 = 36Z^2 - Y^2 \tag{7}$$

Factorizing (7) we have

$$(X + 2Z)(X - 2Z) = (6Z + Y)(6Z - Y) \tag{8}$$

Which is equivalent to the system of double equations

$$\left.\begin{aligned} BX - AY + 2Z(B - 3A) = 0 \\ AX + BY - 2Z(A + 3B) = 0 \end{aligned}\right\} \tag{9}$$

Applying the method of cross multiplication, we get

$$\left.\begin{aligned} X = X(A,B) = 2A^2 + 12AB - 2B^2 \\ Y = Y(A,B) = -6A^2 + 6B^2 + 4AB \\ Z = Z(A,B) = A^2 + B^2 \end{aligned}\right\} \tag{10}$$

Thus (10) represents non-zero distinct integer solutions of (1) in two parameters.

**Properties**

- $X(A+2, A+2) - T_{26,A} \equiv 59 \pmod{48}$
- $Y(3,2A) - T_{50,A} \equiv -7 \pmod{47}$
- $X(A, A+1) - 12\,\mathrm{Pr}_A \equiv -2 \pmod{4}$
- $X(A, 2A^2+1) - 36OH_A + 8T_{4,A}^2 + 6T_{4,A} \equiv 0 \pmod{2}$
- $X(A, A+1) - T_{26,A} \equiv -2 \pmod{19}$

**2.4 PATTERN: 4**

Also (8) is equivalent to the following two equations

$$\left.\begin{aligned} BX - AY - 2Z(B + 3A) = 0 \\ -AX - BY + 2Z(3B - A) = 0 \end{aligned}\right\} \tag{11}$$

Repeating the process as in pattern:3 , the corresponding non-zero distinct integer solution of (1) are given by

$$\left.\begin{aligned} X = X(A,B) = 2A^2 + 2B^2 \\ Y = Y(A,B) = 6A^2 - 6B^2 + 4AB \\ Z = Z(A,B) = -A^2 - B^2 \end{aligned}\right\} \tag{12}$$

Thus (12) represents non-zero distinct integer solutions of (1)

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**Properties:**

- $Y(A,-2)-T_{14,A}+A=-24$
- $X(A,1)-T_{6,A}\equiv 0 \pmod 3$
- $Y(3B,2)-T_{50,B}\equiv -24 \pmod{39}$
- $Y(A,1)-T_{14,A}\equiv -6 \pmod 9$
- $Y(A,2)-T_{14,A}\equiv -11 \pmod{13}$
- $Y(A,A+1)-4\Pr_A\equiv -6 \pmod{12}$

**2.5 PATTERN: 5**

The ternary quadratic equation (1) can be written as

$$X^2=40Z^2-Y^2 \tag{13}$$

Assume $X=X(A,B)=40A^2-B^2$ ; A, B>0 (14)

Substituting (14) in (1) and employing the method of factorization, define

$$(2\sqrt{10}A+B)^2(2\sqrt{10}A-B)^2=(2\sqrt{10}Z+Y)(2\sqrt{10}Z-Y)$$

Equating rational and irrational factors, we get

$$\left.\begin{aligned} Y=Y(A,B)=40A^2+B^2 \\ Z=Z(A,B)=2AB \end{aligned}\right\} \tag{15}$$

Thus (14) and (15) represents non-zero distinct integer solutions of (1) in two parameters

**Properties**:

- $X(A,B)+Y(A,B)=Z(8A,5A)$
- $Y(1,B)-2Z(1,B)-T_{10,A}+B=40$
- $X(20A,A)-Y(A,B)=-T_{4,B}$
- $X(A,B)+Y(A,B)=80T_{4,A}$
- $Y(A,A+1)-X(A,2)-T_{4,A}\equiv 5 \pmod 2$

**2.6 PATTERN: 6**

The ternary quadratic equation (1) can be written as

$$40Z^2-Y^2=X^2*1 \tag{16}$$

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Write 1 as

$$1=(\sqrt{10}+\sqrt{3})(\sqrt{10}-\sqrt{3}) \quad (17)$$

Substituting (14) and (17) in (16) and employing the method of factorization, define

$$(2\sqrt{10}Z+Y)(2\sqrt{10}Z-Y)=(2\sqrt{10}A+B)^2(2\sqrt{10}A-B)^2(\sqrt{10}+\sqrt{3})(\sqrt{10}-\sqrt{3}$$

Equating rational and irrational factors, we get

$$\left.\begin{aligned} Y=Y(A,B)&=120A^2+3B^2+40AB \\ Z=Z(A,B)&=\frac{1}{2}[40A^2+B^2+12AB] \end{aligned}\right\} \quad (18)$$

As our interest is on finding integer solutions, we choose A and B suitably so that the values of X, Y and Z are in integers. In what follows, the values of A, B and the corresponding integer solutions are exhibited.

**Case: 1**

Replace A by 2A and B by 2B

The corresponding solutions of (1) in two parameters are

$$X=X(A,B)=160A^2-4B^2$$

$$Y=Y(A,B)=480A^2+12B^2+160AB$$

$$Z=Z(A,B)=80A^2+2B^2+24AB$$

**Properties:**

- $2Z(A,A)-X(A,A)=56T_{4,A}$
- $3X(A,-A)-Y(A,-A)-Z(A,-A)=78T_{4,A}$
- $X(A,3A+2)-T_{250,A}\equiv -16 \pmod{75}$
- $Z(A,2)-T_{162,A}\equiv 8 \pmod{127}$
- $X(A-2,A-2)-T_{314,A}\equiv 155 \pmod{469}$
- $Z(A,A+1)-24\,\mathrm{Pr}_A-T_{6,A}\equiv 2 \pmod{45}$
- $Z(A,2A^2+1)-84T_{4,A}-4T_{4,A}^2-72OH_A\equiv 0 \pmod{2}$

**Case: 2**

Replace A by 2A+1 and B by 2B

The corresponding solutions of (1) in two parameters are

$$X=X(A,B)=160A^2-4B^2+160A+40$$

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$$Y = Y(A,B) = 480A^2 + 12B^2 + 160AB + 480A + 80B + 120$$
$$Z = Z(A,B) = 80A^2 + 2B^2 + 104A + 12B + 20$$

**Properties**

- $X(A,1) - Z(A,1) - T_{162,A} \equiv -2 \pmod{23}$
- $Z(A,A) - T_{162,A} - T_{6,A} \equiv 20 \pmod{196}$
- $X(-1,B) + T_{4,B} \equiv 0 \pmod{40}$
- $Z(A,A) - T_{166,A} \equiv 20 \pmod{197}$
- $X(A,2A) - T_{290,A} \equiv 40 \pmod{303}$

**Case: 3**

Replace B by 2B
The corresponding solutions of (1) in two parameters are

$$X = X(A,B) = 40A^2 - 4B^2$$
$$Y = Y(A,B) = 120A^2 + 12B^2 + 80AB$$
$$Z = Z(A,B) = 20A^2 + 2B^2 + 12AB$$

**Properties:**

- $Z(A,2) - T_{42,A} \equiv 4 \pmod{19}$
- $Y(-1,B) - T_{26,B} \equiv 51 \pmod{69}$
- $X(1,B) - 2Z(1,B) + T_{18,B} \equiv 0 \pmod{19}$
- $Y(A,4) - X(A+5, A+5) - T_{170,A} \equiv -20 \pmod{43}$
- $Z(A,2) - T_{42,A} \equiv 8 \pmod{43}$
- $Z(2,B) - T_{6,B} \equiv 80 \pmod{13}$

**2.7 PATTERN: 7**

Instead of (17), write 1 as

$$1 = \frac{(\sqrt{10}+1)(\sqrt{10}-1)}{9} \qquad (19)$$

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Following the procedure presented in pattern: 6, the corresponding values of X and Y obtained from (1) are

$$\left.\begin{aligned} Y = Y(A,B) = \frac{1}{3}(40A^2 + B^2 + 40AB) \\ Z = Z(A,B) = \frac{1}{6}(40A^2 + B^2 + 4AB) \end{aligned}\right\} \quad (20)$$

As our interest is on finding integer solutions, we choose A and B suitably so that the values of X, Y and Z are integers. Replace A by 6A and B by 6B

The corresponding solutions of (1) in two parameters are

$$X = X(A,B) = (1440A^2 - 36B^2)$$
$$Y = Y(A,B) = (480A^2 + 12B^2 + 480AB)$$
$$Z = Z(A,B) = (240A^2 + 6B^2 + 24AB)$$

**Properties:**

- $X(A,1) - Y(A,1) - Z(A,1) - T_{402,A} - T_{342,A} - T_{382,A} - T_{322,A} \equiv -54 \pmod{212}$
- $X(A,1) - 6Z(A,1) \equiv -72 \pmod{144}$
- $Y(1,B) - T_{26,B} \equiv 480 \pmod{491}$
- $Z[A, A(A+1)] - 240T_{4,A} - 24T_{3,A}^2 = 48P_A^5$

**3. REMARKABLE OBSERVATION:**

**I:** If the non-zero integer triple $(X_0, Y_0, Z_0)$ is any solution of (1) then each of the following two triples of integers also satisfy (1)

**Triple: 1** $(X_0, Y_n, Z_n)$

Where,

$$Y_n = \frac{1}{2}\{[20 - 18(-1)^n]Y_0 - 120[1 - (-1)^n]Z_0\}$$
$$Z_n = \frac{1}{2}\{3[1 - (-1)^n]Y_0 + [-18 + 20(-1)^n]Z_0\}$$

**Triple: 2** $(X_n, Y_0, Z_n)$

Where,

$$X_n = \frac{1}{4}[(40\alpha^n - 36\beta^n)X_0 - 240(\alpha^n - \beta^n)Z_0]$$

$$Z_n = \frac{1}{4}[6(\alpha^n - \beta^n)X_0 + (-36\alpha^n + 40\beta^n)Z_0]$$

In which $\alpha = 2, \beta = -2$

**II:** Employing the solution (X, Y, Z) of (1) each of the following expressions among the special polygonal and pyramidal numbers is a perfect square.

- $\frac{1}{40}\left\{\left(\frac{3P_{X-2}^3}{T_{3,X-2}}\right)^2 + \left(\frac{P_Y^5}{T_{3,Y}}\right)^2\right\}$
- $40\left\{\left(\frac{P_Z^5}{T_{3,Z}}\right)^2 - \left(\frac{3P_{X-2}^3}{T_{3,X-2}}\right)^2\right\}$
- $40\left\{\left(\frac{3P_{Z-2}^3}{T_{3,Z-2}}\right)^2 - \left(\frac{P_X^5}{T_{3,X}}\right)^2\right\}$

It is worth to note that, on multiplying each of the above observation by 6, it represents a nasty number.

## 4. CONCLUSION

In this paper, we have presented six different patterns of non-zero distinct integer solutions of the homogeneous cone given by $X^2 + Y^2 = 40Z^2$.To conclude, one may search for other patterns of solution and their corresponding properties

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