

THE PRODUCT SPAN OF A FINITE SUBSET OF A COMPLETELY BOUNDED ARTEX SPACE OVER A BI-MONOID

ABSTRACT

The product of subsets of an Artex space over a bi-monoid is defined. Product Combination of elements of a Completely Bounded Artex Space over a bi-monoid is defined. Product span of a finite subset of a completely bounded Artex space over a bi-monoid is defined. A proposition is found and proved. Example is provided. The product of subsets of an Artex space over a bi-monoid is defined. Product Combination of elements of a Completely Bounded Artex Space over a bi-monoid is defined. Product span of a finite subset of a completely bounded Artex space over a bi-monoid is defined. A proposition is found and proved. Example is provided.

Keywords: Bi-monoids, Artex Spaces over bi-monoids, Completely Bounded Artex Spaces over bi-monoids, Product Combination, Product Span.

I. INTRODUCTION

There are many areas such as engineering and science to which Boolean algebra is applied. George Boole in 1854 introduced an algebraic system called Boolean Algebra. A more general algebraic system is the lattice. A Boolean Algebra is then introduced as a special lattice. Lattices and Boolean algebra have important applications in the theory and design of computers. As the theory of Artex Spaces over bi-monoids is developed from Lattice theory and linear algebra, theory of Artex Spaces over bi-monoids will be more useful. The algebraic system Bi-semi-group is more general to the existing algebraic system ring or an associative ring. Complete Artex Spaces over bi-monoids, Lower Bounded Artex Spaces over bi-monoids, Upper Bounded Artex Spaces over bi-monoids, Bounded Artex Spaces over bi-monoids and Completely Bounded Artex Spaces over a bi-monoids are very interesting and will be very useful. Product Combination of elements of a Completely Bounded Artex Space over a bi-monoid and Product span of a finite subset of a completely bounded Artex space over a bi-monoid will be useful to form many SubArtex spaces.

II. PRELIMINARIES

2.1 Lattice : A lattice is a partially ordered set (L, \leq) in which every pair of elements $a, b \in L$ has a greatest lower bound and a least upper bound.

The greatest lower bound of a and b is denoted by $a \wedge b$ and the least upper bound of a and b is denoted by $a \vee b$

2.2 Lattice as an Algebraic System : A lattice is an algebraic system (L, \wedge, \vee) with two binary operations \wedge and \vee on L which are both commutative, associative and satisfy the absorption laws namely $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$, for all $a, b \in L$

The operations \wedge and \vee are called cap and cup respectively, or sometimes meet and join respectively.

2.3 Properties : We have the following properties in a lattice (L, \wedge, \vee)

- | | | |
|--|---|--|
| 1. $a \wedge a = a$ | 1'. $a \vee a = a$ | (Idempotent Law) |
| 2. $a \wedge b = b \wedge a$ | 2'. $a \vee b = b \vee a$ | (Commutative Law) |
| 3. $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ | 3'. $(a \vee b) \vee c = a \vee (b \vee c)$ | (Associative Law) |
| 4. $a \wedge (a \vee b) = a$ | 4'. $a \vee (a \wedge b) = a$ | for all $a, b, c \in L$ (Absorption Law) |

2.4 Complete Lattice : A lattice is called a complete lattice if each of its nonempty subsets has a least upper bound and a greatest lower bound.

Every finite lattice is a complete lattice and every complete lattice must have a least element and a greatest element.

The least and the greatest elements, if they exist, are called the bounds or units of the lattice and are denoted by 0 and 1 respectively.

2.5 Bounded Lattice : A lattice which has both elements 0 and 1 is called a bounded lattice. A bounded lattice is denoted by $(L, \wedge, \vee, 0, 1)$

The bounds 0 and 1 of a lattice (L, \wedge, \vee) satisfy the following identities.

For any $a \in L$, $a \vee 0 = a$ $a \wedge 1 = a$ $a \vee 1 = 1$ $a \wedge 0 = 0$

2.5.1 Example : For any set S, the lattice $(P(S), \subseteq)$ is a bounded lattice. Here for each $A, B \in P(S)$, the least upper bound of A and B is $A \cup B$ and the greatest lower bound of A and B is $A \cap B$. The bounds in this lattice are ϕ , the empty set and S, the universal set.

2.6 Complemented Lattice : Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice. An element $a' \in L$ is called a complement of an element $a \in L$ if $a \wedge a' = 0$, $a \vee a' = 1$. A bounded lattice $(L, \wedge, \vee, 0, 1)$ is said to be a complemented lattice if every element of L has at least one complement. A complemented lattice is denoted by $(L, \wedge, \vee, ', 0, 1)$.

2.6.1 Example : For any set S, the lattice $(P(S), \subseteq)$ is a Complemented lattice.

For each $A, B \in P(S)$, the least upper bound of A and B is $A \cup B$ and the greatest lower bound of A and B is $A \cap B$.

The bounds in this lattice are ϕ , the empty set and S, the universal set.

Here for any $A \in P(S)$, the complement of A in P(S) is S-A

2.7 Doubly Closed Space: A non-empty set D together with two binary operations denoted by + and . is called a Doubly Closed Space if (i) $a.(b+c) = a.b + a.c$ and (ii) $(a+b).c = a.c + b.c$, for all $a, b, c \in D$

A Doubly closed space is denoted by $(D, +, .)$

Note 1: The axioms (i) $a.(b+c) = a.b + a.c$ and (ii) $(a+b).c = a.c + b.c$, for all $a, b, c \in D$ are called the distributive properties of the Doubly Closed Space.

Note 2: The operations + and . need not be the usual addition and usual multiplication respectively.

2.7.1 Example : Let N be the set of all natural numbers.

Then $(N, +, .)$, where + is the usual addition and . is the usual multiplication, is a Doubly closed space.

Similarly $(Z, +, .)$, $(Q, +, .)$, $(R, +, .)$ and $(C, +, .)$ are all Doubly closed spaces.

2.7.2 Example : $(Z, +, -)$, where + is the usual addition and - is the usual subtraction, is not a Doubly closed space.

Even though + and - are binary operations in Z, $(Z, +, -)$ is not a Doubly closed space because of the distributive properties of the Doubly Closed Space.

Take $a = 15$, $b = 7$, $c = 4$

Then $a - (b + c) = 15 - (7 + 4)$

$$= 15 - 11$$

$$= 4$$

But $(a - b) + (a - c) = (15 - 7) + (15 - 4)$

$$= 8 + 11$$

$$= 19$$

Therefore, $a - (b + c) \neq (a - b) + (a - c)$

Therefore, $(Z, +, -)$ is not a Doubly closed space.

2.8 Bi-semi-group : A Doubly closed space $(S, +, .)$ is called a Bi-semi-group if + and . are associative in D.

2.8.1Example : $(\mathbb{N}, +, \cdot), (\mathbb{Z}, +, \cdot), (\mathbb{Q}, +, \cdot), (\mathbb{R}, +, \cdot),$ and $(\mathbb{C}, +, \cdot),$ where $+$ is the usual addition and \cdot is the usual multiplication, are all Bi-semi-groups.

2.9Bi-monoid : A Bi-semi-group $(M, +, \cdot)$ is called a Bi-monoid if there exist elements denoted by 0 and 1 in S such that $a+0=a=0+a,$ for all $a \in M$ and $a \cdot 1=a=1 \cdot a,$ for all $a \in M$.

The element 0 is called the identity element of M with respect to the binary operation $+$ and the element 1 is called the identity element of M with respect to the binary operation..

2.9.1Example : Let $W = \{0,1,2,3,\dots\}.$ Then $(W, +, \cdot),$ where $+$ is the usual addition and \cdot is the usual multiplication, is a Bi-monoid.

2.9.2Example : Let $Q^+ = \mathbb{Q}^+ \cup \{0\},$ where \mathbb{Q}^+ is the set of all positive rational numbers. Then $(Q^+, +, \cdot)$ is a bi-monoid.

2.9.3Example : $\mathbb{R}^+ = \mathbb{R}^+ \cup \{0\},$ where \mathbb{R}^+ is the set of all positive real numbers. Then $(\mathbb{R}^+, +, \cdot)$ is a bi-monoid.

2.9.10Example : $(\mathbb{Z}, +, \cdot), (\mathbb{Q}, +, \cdot), (\mathbb{R}, +, \cdot),$ and $(\mathbb{C}, +, \cdot),$ where $+$ is the usual addition and \cdot is the usual multiplication, are all Bi-monoids.

2.10 Artex Space Over a Bi-monoid : Let $(M, +, \cdot)$ be a bi-monoid with the identity elements 0 and 1 with respect to $+$ and \cdot respectively. A non-empty set A together with two binary operations \wedge and \vee is said to be an Artex Space Over the Bi-monoid $(M, +, \cdot)$ if

1. (A, \wedge, \vee) is a lattice and

2. for each $m \in M, m \neq 0,$ and $a \in A,$ there exists an element $ma \in A$ satisfying the following conditions :

(i) $m(a \wedge b) = ma \wedge mb$

(ii) $m(a \vee b) = ma \vee mb$

(iii) $ma \wedge na \leq (m+n)a$ and $ma \vee na \leq (m+n)a$

(iv) $(mn)a = m(na),$ for all $m, n \in M, m \neq 0, n \neq 0,$ and $a, b \in A$

(v) $1 \cdot a = a,$ for all $a \in A.$

Here, \leq is the partial order relation corresponding to the lattice $(A, \wedge, \vee).$ The multiplication ma is called a **bi-monoid multiplication with an artex element** or simply bi-monoid multiplication in A.

2.10.1 Example : Let $W = \{0,1,2,3,\dots\}.$

Then $(W, +, \cdot)$ is a bi-monoid, where $+$ and \cdot are the usual addition and multiplication respectively.

Let Z be the set of all integers

Then (Z, \leq) is a lattice in which \wedge and \vee are defined by $a \wedge b = \text{minimum of } \{a,b\}$ and $a \vee b = \text{maximum of } \{a,b\},$ for all $a, b \in Z.$

Clearly for each $m \in W, m \neq 0,$ and for each $a \in Z, ma \in Z.$

Also

(i) $m(a \wedge b) = ma \wedge mb$

(ii) $m(a \vee b) = ma \vee mb$

(iii) $ma \wedge na \leq (m+n)a$ and $ma \vee na \leq (m+n)a$

(iv) $(mn)a = m(na)$

(v) $1 \cdot a = a,$ for all $m, n \in W, m \neq 0, n \neq 0$ and $a, b \in Z$

Therefore, Z is an Artex Space Over the Bi-monoid $(W, +, \cdot)$

Properties 2.11.1 : We have the following properties in a lattice (L, \wedge, \vee)

1. $a \wedge a = a$

1'. $a \vee a = a$

2. $a \wedge b = b \wedge a$

2'. $a \vee b = b \vee a$

3. $(a \wedge b) \wedge c = a \wedge (b \wedge c)$

3'. $(a \vee b) \vee c = a \vee (b \vee c)$

4. $a \wedge (a \vee b) = a$

4'. $a \vee (a \wedge b) = a,$ for all $a, b, c \in L$

Therefore, we have the following properties in an Artex Space A over a bi-monoid M .

- | | | | |
|-------|---|--------|---|
| (i) | $m(a \wedge a) = ma$ | (i)' | $m(a \vee a) = ma$ |
| (ii) | $m(a \wedge b) = m(b \wedge a)$ | (ii)' | $m(a \vee b) = m(b \vee a)$ |
| (iii) | $m((a \wedge b) \wedge c) = m(a \wedge (b \wedge c))$ | (iii)' | $m((a \vee b) \vee c) = m(a \vee (b \vee c))$ |
| (iv) | $m(a \wedge (a \vee b)) = ma$ | (iv)' | $m(a \vee (a \wedge b)) = ma,$ |

for all $m \in M, m \neq 0$ and $a, b, c \in A$

2.12 SubArtex Space : Let (A, \wedge, \vee) be an Artex space over a bi-monoid. $(M, +, \cdot)$. Let S be a nonempty subset of A . Then S is said to be a SubArtex Space of A if (S, \wedge, \vee) itself is an Artex Space over M .

2.12.1 Example : As defined in Example 3.2.1, Z is an Artex Space over $W = \{0, 1, 2, 3, \dots\}$ and W is a subset of Z . Also W itself is an Artex space over W under the operations defined in Z . Therefore, W is a SubArtex space of Z .

2.13 Complete Artex Space over a bi-monoid : An Artex space A over a bi-monoid M is said to be a Complete Artex Space over M if as a lattice, A is a complete lattice, that is each nonempty subset of A has a least upper bound and a greatest lower bound.

2.13.1 Remark : Every Complete Artex space must have a least element and a greatest element.

The least and the greatest elements, if they exist, are called the bounds or units of the Artex space and are denoted by 0 and 1 respectively.

2.14 Lower Bounded Artex Space over a bi-monoid : An Artex space A over a bi-monoid M is said to be a Lower Bounded Artex Space over M if as a lattice, A has the least element 0 .

2.14.1 Example : Let A be the set of all constant sequences (x_n) in $[0, \infty)$

Let $W = \{0, 1, 2, 3, \dots\}$.

Define \leq' , an order relation, on A by for $(x_n), (y_n)$ in A , $(x_n) \leq' (y_n)$ means $x_n \leq y_n$, for each n where \leq is the usual relation "less than or equal to"

Therefore, A is an Artex space over W .

The sequence (0_n) , where 0_n is 0 for all n , is a constant sequence belonging to A

Also $(0_n) \leq' (x_n)$, for all the sequences (x_n) belonging to in A

Therefore, (0_n) is the least element of A .

That is, the sequence $0, 0, 0, \dots$ is the least element of A

Hence A is a Lower Bounded Artex space over W .

2.15 Upper Bounded Artex Space over a bi-monoid : An Artex space A over a bi-monoid M is said to be an Upper Bounded Artex Space over M if as a lattice, A has the greatest element 1 .

2.15.1 Example : Let A be the set of all constant sequences (x_n) in $(-\infty, 0]$ and let $W = \{0, 1, 2, 3, \dots\}$.

Define \leq' , an order relation, on A by for $(x_n), (y_n)$ in A , $(x_n) \leq' (y_n)$ means $x_n \leq y_n$, for $n = 1, 2, 3, \dots$, where \leq is the usual relation "less than or equal to"

A is an Artex space over W .

Now, the sequence (1_n) , where 1_n is 0 , for all n , is a constant sequence belonging to A

Also $(x_n) \leq' (1_n)$, for all the sequences (x_n) in A

Therefore, (1_n) is the greatest element of A .

That is, the sequence $0, 0, 0, \dots$ is the greatest element of A

Hence A is an Upper Bounded Artex Space over W .

2.16 Bounded Artex Space over a bi-monoid : An Artex space A over a bi-monoid M is said to be a Bounded Artex Space over M if A is both a Lower bounded Artex Space over M and an Upper bounded Artex Space over M .

2.16.1 Completely Bounded Artex Space over a bi-monoid: A Bounded Artex Space A over a bi-monoid M is said to be a Completely Bounded Artex Space over M if (i) $0.a = 0$, for all $a \in A$ (ii) $m.0 = 0$, for all $m \in M$.

2.16.2 Note : While the least and the greatest elements of the Complemented Artex Space is denoted by 0 and 1 , the identity elements of the bi-monoid $(M, +, \cdot)$ with respect to addition and multiplication are, if no confusion arises, also denoted by 0 and 1 respectively.

III. THE PRODUCT OF SUBSETS OF AN ARTEX SPACE OVER A BI-MONOID

3.1 THE PRODUCT OF SUBSETS OF AN ARTEX SPACE OVER A BI-MONOID : Let (A, \wedge, V) be an Artex Space over a bi-monoid $(M, +, \cdot)$. Let S and T be subsets of the Artex Space A . Then the product of S and T denoted by $S \wedge T$ is defined by $S \wedge T = \{ s \wedge t / s \in S \text{ and } t \in T \}$

IV. THE PRODUCT SPAN OF A SUBSET OF A COMPLETELY BOUNDED ARTEX SPACE OVER A BI-MONOID

4.1 Product Combination : Let (A, \wedge, V) be a Completely Bounded Artex Space over a bi-monoid $(M, +, \cdot)$. Let $a_1, a_2, a_3, \dots, a_n \in A$. Then any element of the form $m_1 a_1 \wedge m_2 a_2 \wedge m_3 a_3 \wedge \dots \wedge m_n a_n$, where $m_i \in M$, is called a Product Combination or Meet Combination of $a_1, a_2, a_3, \dots, a_n$ over the Artex Space A .

4.2 The Product Span of a subset of a Completely Bounded Artex Space over a Bi-monoid : Let (A, \wedge, V) be a Completely Bounded Artex Space over a bi-monoid $(M, +, \cdot)$ and W be a nonempty finite subset of A . Then the Product Span of W or Meet Span of W denoted by $P[W]$ is defined to be the set of all product combinations of elements of W . That is, if $W = \{ a_1, a_2, a_3, \dots, a_n \}$, then $P[W] = \{ m_1 a_1 \wedge m_2 a_2 \wedge m_3 a_3 \wedge \dots \wedge m_n a_n / m_i \in M \}$.

PROPOSITION : 4.3.1 : Let (A, \wedge, V) be a Completely Bounded Artex Space over a bi-monoid $(M, +, \cdot)$. Let W and V be any two nonempty finite subsets of A . Then $P[W \cup V] = P[W] \wedge P[V]$.

Proof : Let (A, \wedge, V) be a Completely Bounded Artex Space over a bi-monoid $(M, +, \cdot)$

Let W and V be any two nonempty finite subsets of A such that $W = \{ a_1, a_2, a_3, \dots, a_n \}$ and $V = \{ b_1, b_2, b_3, \dots, b_k \}$

Let $x \in P[W \cup V]$

Since \wedge is commutative and associative, we may arrange x as

$x = m_1 a_1 \wedge m_2 a_2 \wedge m_3 a_3 \wedge \dots \wedge m_n a_n \wedge m_{n+1} b_1 \wedge m_{n+2} b_2 \wedge \dots \wedge m_{n+k} b_k$, where $m_i \in M$

Let $w = m_1 a_1 \wedge m_2 a_2 \wedge m_3 a_3 \wedge \dots \wedge m_n a_n$ and let $v = m_{n+1} b_1 \wedge m_{n+2} b_2 \wedge \dots \wedge m_{n+k} b_k$

Then $x = w \wedge v$

Clearly $w = m_1a_1 \wedge m_2a_2 \wedge m_3a_3 \wedge \dots \wedge m_na_n \in P[W]$ and

$$v = m_{n+1}b_1 \wedge m_{n+2}b_2 \wedge \dots \wedge m_{n+k}b_k \in P[V]$$

Therefore, $x = w \wedge v \in P[W] \wedge P[V]$

Therefore, $P[W \cup V] \subseteq P[W] \wedge P[V]$ ----- (i)

Conversely, let $x \in P[W] \wedge P[V]$

Then $x = w \wedge v$, where $w \in P[W]$ and $v \in P[V]$

Therefore, $w = m_1a_1 \wedge m_2a_2 \wedge m_3a_3 \wedge \dots \wedge m_na_n$, $m_i \in M$ and

$$v = m_{n+1}b_1 \wedge m_{n+2}b_2 \wedge \dots \wedge m_{n+k}b_k, m_i \in M$$

Then $x = w \wedge v$

$$x = m_1a_1 \wedge m_2a_2 \wedge m_3a_3 \wedge \dots \wedge m_na_n \wedge m_{n+1}b_1 \wedge m_{n+2}b_2 \wedge \dots \wedge m_{n+k}b_k, \text{ where } m_i \in M$$

Therefore, $x \in P[W \cup V]$

Therefore, $P[W] \wedge P[V] \subseteq P[W \cup V]$ ----- (ii)

From (i) and (ii) we have $P[W \cup V] = P[W] \wedge P[V]$.

4.4Example : Let $R' = R^+ \cup \{0\}$, where R^+ is the set of all positive real numbers and let $W = \{0,1,2,3,\dots\}$
 (R', \leq) is a lattice in which \wedge and \vee are defined by $a \wedge b = \min\{a,b\}$ and $a \vee b = \max\{a,b\}$, for all $a,b \in R'$.

Here ma is the usual multiplication of a by m .

Clearly for each $m \in W, m \neq 0$, and for each $a \in R'$, $ma \in R'$.

Also,

(i) $m(a \wedge b) = ma \wedge mb$

(ii) $m(a \vee b) = ma \vee mb$

(iii) $ma \wedge na \leq (m+n)a$ and $ma \vee na \leq (m+n)a$

(iv) $(mn)a = m(na)$, for all $m,n \in W, m \neq 0, n \neq 0$, and $a,b \in R'$

(v) $1.a = a$, for all $a \in R'$

Therefore, R' is an Artex Space Over the bi-monoid $(W, +, \cdot)$

Generally, if Λ_1, Λ_2 , and Λ_3 are the cap operations of A, B and C respectively and if \vee_1, \vee_2 , and \vee_3 are the cup operations of A, B and C respectively, then the cap of $A \times B \times C$ denoted by Λ and the cup of $A \times B \times C$ denoted by \vee are defined

$$x \wedge y = (a_1, b_1, c_1) \wedge (a_2, b_2, c_2) = (a_1 \wedge_1 a_2 \wedge_1 a_3, b_1 \wedge_2 b_2 \wedge_2 b_3, c_1 \wedge_3 c_2 \wedge_3 c_3) \text{ and}$$

$$x \vee y = (a_1, b_1, c_1) \vee (a_2, b_2, c_2) = (a_1 \vee_1 a_2 \vee_1 a_3, b_1 \vee_2 b_2 \vee_2 b_3, c_1 \vee_3 c_2 \vee_3 c_3)$$

Here, $\Lambda_1, \Lambda_2,$ and Λ_3 denote the same meaning minimum of two elements in R' and $\vee_1, \vee_2,$ and \vee_3 denote the same meaning maximum of two elements in R'

Therefore, $R'^3 = R' \times R' \times R'$ is an Artex Space over W , where cap and cup operations are denoted by Λ and \vee respectively.

Let $H = \{ (1,0,0) \}$ and let $T = \{ (0,1,0) \}$

Now $P[H] = \{ (m,0,0) / m \in R' \}$ and $P[T] = \{ (0,n,0) / n \in R' \}$

$$P[H] \Lambda P[T] = \{ (m,0,0) / m \in R' \} \vee \{ (0,n,0) / n \in R' \}$$

$$= \{ (m \Lambda_1 0, 0 \Lambda_2 n, 0 \Lambda_3 0) \}$$

$$= \{ (0,0,0) \} \text{ (since } m \Lambda_1 0 = \text{mini.} \{m,0\} = 0, 0 \Lambda_2 n = \text{mini.} \{0,n\} = 0 \text{ and } 0 \Lambda_3 0 = \text{mini.} \{0,0\} = 0)$$

$$P[H] \Lambda P[T] = \{ (0,0,0) \} \text{ ----- (i)}$$

Now $H \cup T = \{ (1,0,0), (0,1,0) \}$

Let $m, n \in M, m \neq 0, n \neq 0$

$$\text{Then } m(1,0,0) \Lambda n(0,1,0) = (m,0,0) \Lambda (0,n,0)$$

$$= (m \Lambda_1 0, 0 \Lambda_2 n, \Lambda_3 0)$$

$$= (0,0,0) \text{ (since } m \Lambda_1 0 = \text{mini.} \{m,0\} = 0, 0 \Lambda_2 n = \text{mini.} \{0,n\} = 0 \text{ and}$$

$$0 \Lambda_3 0 = \text{mini.} \{0,0\} = 0)$$

$$\text{Therefore, } P[HUT] = \{ (0,0,0) \} \text{ ----- (ii)}$$

From equations (i) and (ii) we have $P[HUT] = P[H] \Lambda P[T]$

V. CONCLUSION

From ArtexSpace to this Product Span we have come across so many miles. Product Combination of elements of an Artex Space over a bi-monoid, Product Span of a finite subset of a completely bounded artex space over a bi-monoid will create a new dimension in the theory of Artex spaces over bi-monoids. Interested researcher can do their research in product span.

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